LOGICISM

What do you want from me!?

And the more important questions are what parts of this reading must I have a really good picture of in order to move forward comfortably?

—Wes

y tho

What I do understand is the concrete absurdity that logicism relays on. While yes, many of these mathematical statements seem to be true and most be in order to further understanding, yet the actual analysis and purpose of these statements is unclear. …

—Jay

Soames is underselling logicism

Though I have plenty of the usual complaints about Russelian arithmetic logic, I have far more with some of Soames's treatment of the subject, particularly on the last few pages. For one thing, I think he could have done a better job explaining at least some of the motivation for the project. Until Frege and Russel, philosophers and mathematicians had (imo) pretty awful conceptions of what a number is. For example, a lot of people bought into Mill's theory that a number is something which impresses the senses in a certain way. Whatever a number is, I'm pretty confident that its not that. Frege had one of the all-time great philosophical takedowns of that theory. But it was only with Rusell's logical reduction approach that we start to get really promising ways to deal with many sorts of numbers. All more modern advancements in proof theory have a lot more in common with Principia than anything that came before it. Even if you want to say much of the rest was a failure (which would also be wrong), that definitely advanced mathematics and was essential for developing really important branches of math. So yeah, I think Soames is incorrect that a) people have always had a good handle on what arithmetic is and how it can be justified and b) it did in fact take a genius like Frege or Russel to get a better way into understanding what "4" is, let alone "4+4".

—Theo

What are mathematical objects?

We seem to know many mathematical truths, and these truths seem to be about mathematical objects. E.g.:

numbers

functions

sets

etc.

So…what are those things?

Abstract Entities
They are non-mental, non-physical objects, existing eternally and located outside space and time.

Okay, but then how can we know about them?

What are mathematical objects?

We seem to know many mathematical truths, and these truths seem to be about mathematical objects. E.g.:

numbers

functions

sets

etc.

So…what are those things?

Ordinary Physical Things
They exist right here in the physical world with us. Maybe they're ordinary things, or maybe they're patterns or structures of those things.

Um…where? Can you show me one?

And what about infinite sets, 5-dimensional spaces, perfect circles and triangles, etc.?

What are mathematical objects?

We seem to know many mathematical truths, and these truths seem to be about mathematical objects. E.g.:

numbers

functions

sets

etc.

So…what are those things?

Mental/linguistic objects
Mathematical objects exist in our minds, or maybe they're things that arise from the conventions governing language.

But why do we seem to have so much objective knowledge about them?

And why is math such a good way to know about the world outside our minds?

Puzzling Facts about Mathematical Truths

Necessity
It's not just a matter of fact that 2+2=4. It apparently could not have been otherwise.

A priority
We seem to be able to know mathematical truths in a pure, not wholly sensory way.

Applicability
Mathematics is a very useful guide to how the world actually works.

Why A Priori?

From what I gather, Russell's attempt to reduce higher mathematics to arithmetic and arithmetics to logic is slightly justifiable in that a lot of the mathematics studied is actually derived from natural scenarios. However, the idea that all high mathematics is priori does not seem to be true.

—Darcy

Why A Priori?

I want an example of our apriori knowledge of mathematical propositions that is independent of experience. I was taking this to mean perhaps perceiving quantities or predicting the trajectory of a ball coming towards you- but these are both observable and easily traced to our sense data, so I must be wrong. Towards the end when Soames started talking about the mathematical proposition of (3x3)+(4x4)=(5x5), he seemed to suggest that this was an example of the math we know independent of experience. But, students need to learn math in school before they know anything about multiplication or mathematical propositions that can be written down, so none of that is apriori or independent of experience.

—Laszlo

Puzzling Facts about Mathematical Truths

Necessity
It's not just a matter of fact that 2+2=4. It apparently could not have been otherwise.

A priority
We seem to be able to know mathematical truths in a pure, not wholly sensory way.

Applicability
Mathematics is a very useful guide to how the world actually works.

Puzzling Facts about Mathematical Truths

Necessity
It's not just a matter of fact that 2+2=4. It apparently could not have been otherwise.

A priority
We seem to be able to know mathematical truths in a pure, not wholly sensory way.

Applicability
Mathematics is a very useful guide to how the world actually works.

Theories about how we know these facts

Empiricism
We learn some mathematical truths using our senses and then generalize like crazy. Mathematical truths are just like other empirical, scientific truths about the regular old physical world.

Puzzling Facts about Mathematical Truths

Necessity
It's not just a matter of fact that 2+2=4. It apparently could not have been otherwise. ❌

A priority
We seem to be able to know mathematical truths in a pure, not wholly sensory way. ❌

Applicability
Mathematics is a very useful guide to how the world actually works. ✅

Theories about how we know these facts

Empiricism
We learn some mathematical truths using our senses and then generalize like crazy. Mathematical truths are just like other empirical, scientific truths about the regular old physical world.

What about our knowledge of perfect circles, infinite sets, 20-dimensional spaces, etc?

And how does all this generalizing work, by the way?

Puzzling Facts about Mathematical Truths

Necessity
It's not just a matter of fact that 2+2=4. It apparently could not have been otherwise.

A priority
We seem to be able to know mathematical truths in a pure, not wholly sensory way.

Applicability
Mathematics is a very useful guide to how the world actually works.

Theories about how we know these facts

Rationalism
We have a priori knowledge of mathematical truths. Maybe it's innate, or derivable using innate reasoning abilities from innate knowledge.

Puzzling Facts about Mathematical Truths

Necessity
It's not just a matter of fact that 2+2=4. It apparently could not have been otherwise. ❌

A priority
We seem to be able to know mathematical truths in a pure, not wholly sensory way. ✅

Applicability
Mathematics is a very useful guide to how the world actually works. ❌

Theories about how we know these facts

Rationalism
We have a priori knowledge of mathematical truths. Maybe it's innate, or derivable using innate reasoning abilities from innate knowledge.

Why would a bunch of primates have evolved to have innate knowledge of abstract pure math?

And why would this knowledge turn out to be so useful for describing the physical world?

Puzzling Facts about Mathematical Truths

Necessity
It's not just a matter of fact that 2+2=4. It apparently could not have been otherwise.

A priority
We seem to be able to know mathematical truths in a pure, not wholly sensory way.

Applicability
Mathematics is a very useful guide to how the world actually works.

Theories about how we know these facts

Kant
Mathematical knowledge is "synthetic a priori" knowledge of our pure intuitions of space and time.

Basically: it's knowledge of the basic structures that we impose on perceptual experience to make it intelligible

Puzzling Facts about Mathematical Truths

Necessity
It's not just a matter of fact that 2+2=4. It apparently could not have been otherwise.

A priority
We seem to be able to know mathematical truths in a pure, not wholly sensory way.

Applicability
Mathematics is a very useful guide to how the world actually works.

Theories about how we know these facts

Kant
Mathematical knowledge is "synthetic a priori" knowledge of our pure intuitions of space and time.

Basically: it's knowledge of the basic structures that we impose on perceptual experience to make it intelligible

Unfortunately, Kant was wrong about which geometry correctly described space. So how could that be a priori?

The Logicist Project

(1) Formulate a logical system capable of capturing the validity of a wide range of valid inferences.

(2) Give analyses (definitions) of all of our basic mathematical concepts (the numbers, addition, subtraction, etc.) in the vocabulary of our new logic.

(3) Formulate some basic logical axioms—self-evident logical truths.

(4) Taking the definitions in (2) and the axioms in (3) as premises, use the logical system from (1) to prove all of the truths of arithmetic as theorems.

Isn’t math easier to grasp than logic?

At the top of 160 Soames finally said what I had been thinking the whole time, which is that Russell's logical axioms are in need of justification more so than arithmetic. I took Russell's attempt to justify mathematics logically as more of an attempt at proving the validity of logic than the validity of math. I fail to see how our sense perceptions don't provide justification for our apriori knowledge of mathematics.

—Laszlo

The Logicist Project

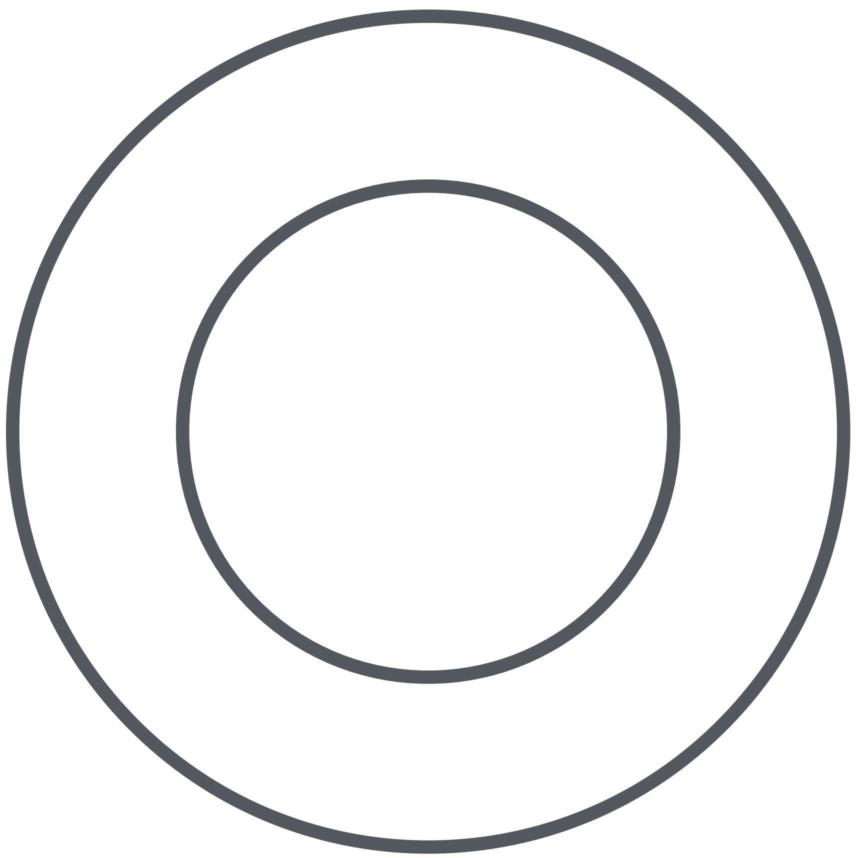
(1) Formulate a logical system capable of capturing the validity of a wide range of valid inferences.

(2) Give analyses (definitions) of all of our basic mathematical concepts (the numbers, addition, subtraction, etc.) in the vocabulary of our new logic.

(3) Formulate some basic logical axioms—self-evident logical truths.

(4) Taking the definitions in (2) and the axioms in (3) as premises, use the logical system from (1) to prove all of the truths of arithmetic as theorems.

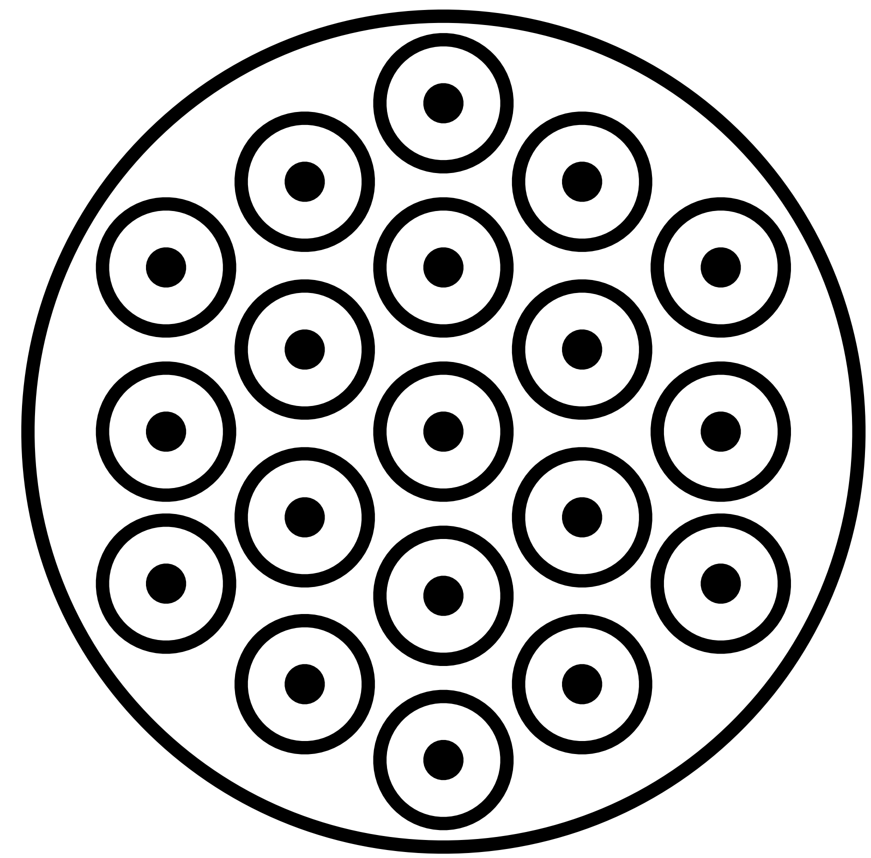
0



0 is the set of all sets
containing no members

$$0 =_{df} \{s : \neg(\exists x)(x \in s)\}$$

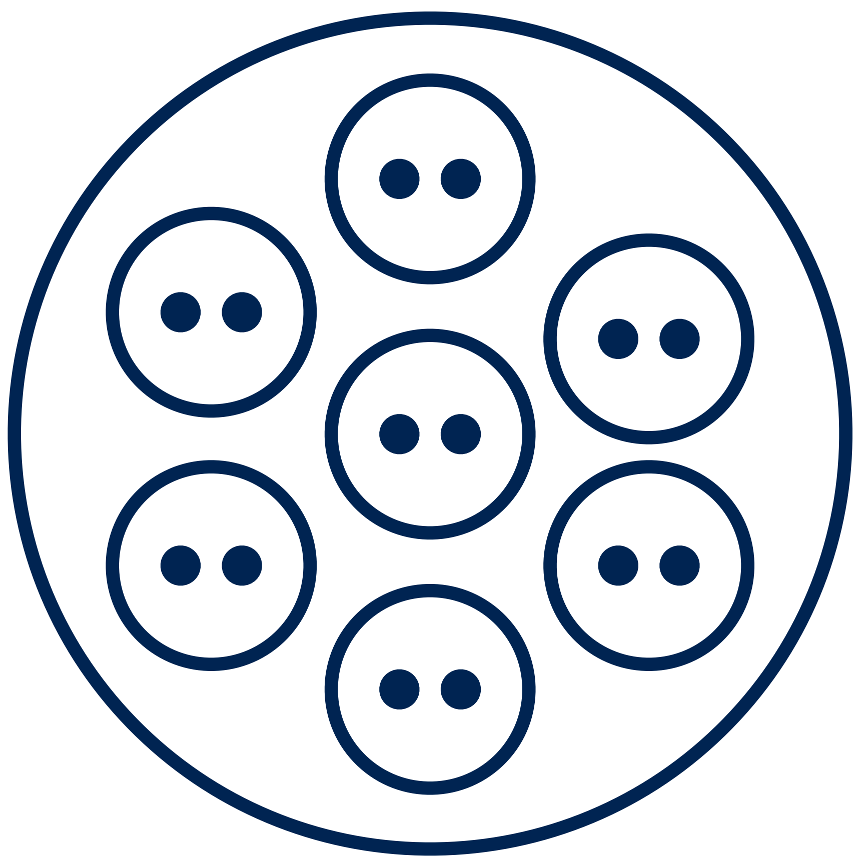
1



1 is the set of all sets
containing a single member

$$1 =_{df} \{s : (\exists x)(x \in s \; \& \; (\forall y)(y \in s \supset y = x))\}$$

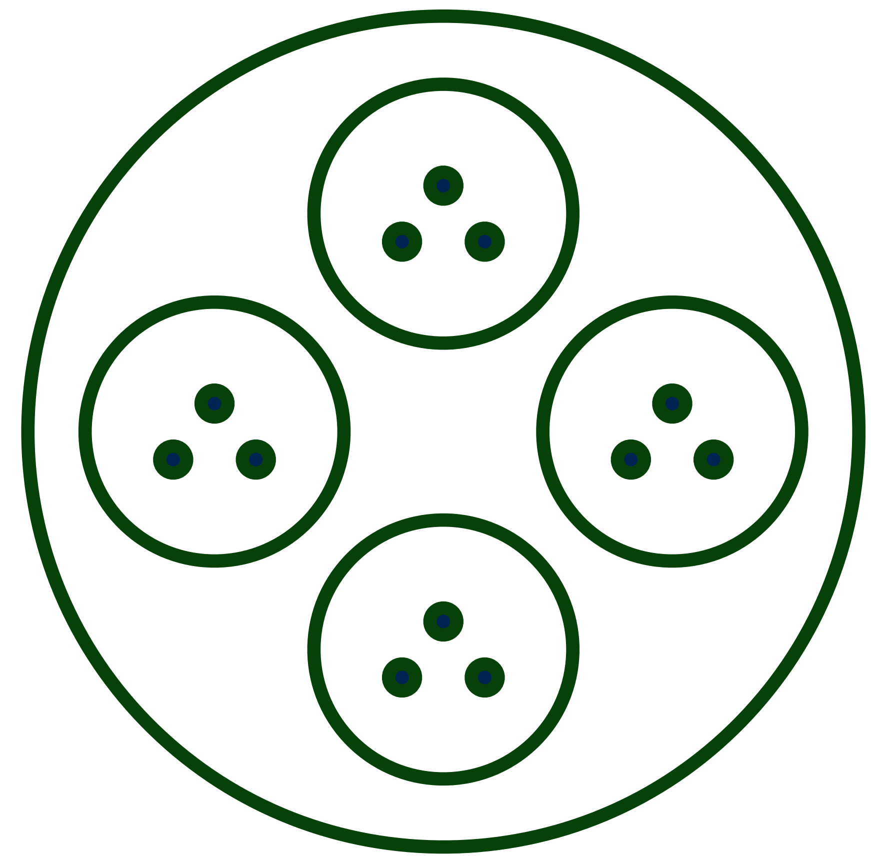
2



2 is the set of all sets that have exactly two members

$$=_{df} \{s : (\exists x)(\exists y)(x \in s \;\&\; y \in s \;\&\; x \neq y \;\&\; (\forall z)(z \in s \supset (z \neq x \supset z = y))\}$$

3



3 is the set of all sets containing exactly three members

$$=_{df} \{s : (\exists x)(\exists y)(\exists z)(x \in s \;\&\; y \in s \;\&\; z \in s \;\&\; x \neq y \;\&\; x \neq z \;\&\; y \neq z \;\&\; (\forall w)(w \in s \supset (w \neq x \supset (w \neq y \supset w = z))))\}$$

PEANO POSTULATES

(1) 0 is a number

(2) The successor of any number is a number

(3) No two numbers have the same successor

(4) 0 is not the successor of any number

(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.

ADDITION

(i) $a + 0 = a$

(ii) $a + S(b) = S(a+b)$

$a + 1$	$= a + S(0)$	(by the def. of 1)
	$= S(a + 0)$	(by (ii))
	$= S(a)$	(by (i))

$a + 2$	$= a + S(1)$	(by the def. of 2)
	$= S(a + 1)$	(by (ii))
	$= S(S(a))$	(by the result of a+1)

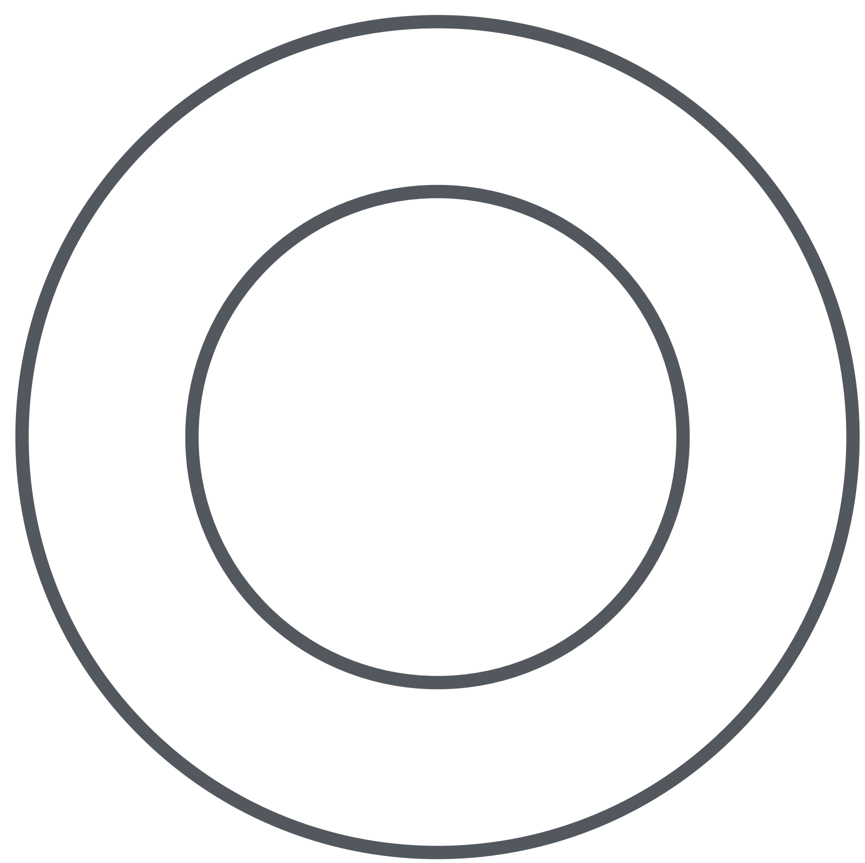
The number that belongs to the concept F = the number that belongs to the concept G

iff

There is a one-to-one correspondence between the Fs and the Gs.

0 = the extension of the concept: is equinumerous with the concept: is not self-identical

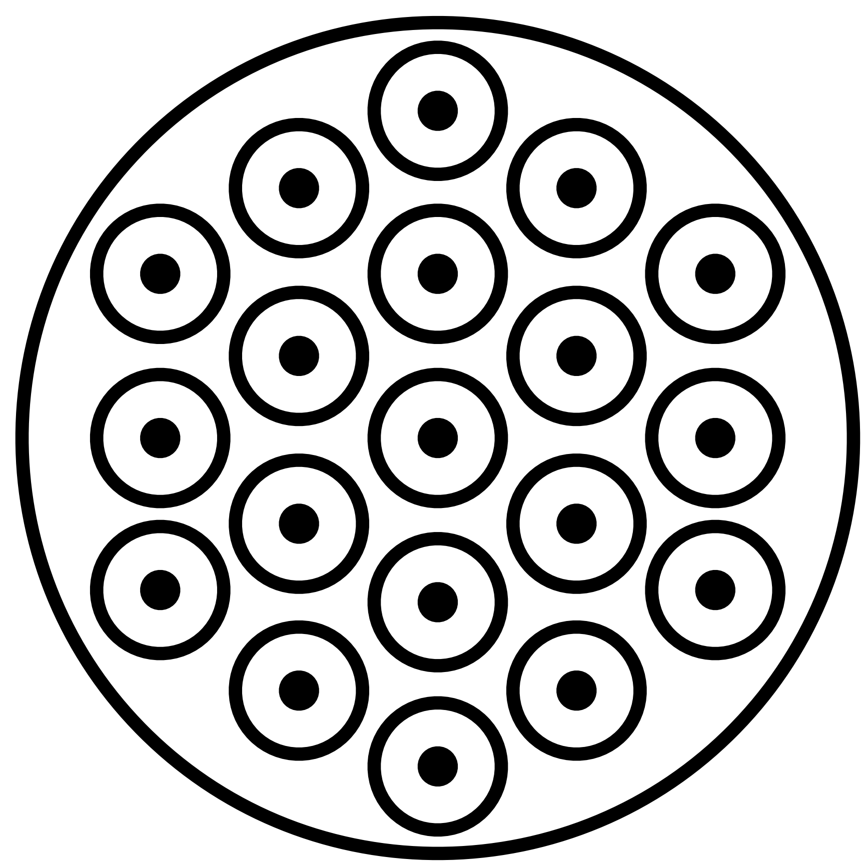
$0 = \{s : s \text{ is equinumerous with } \{x : x \neq x\}\}$



1 = the extension of the concept: is equinumerous with the concept: is identical to 0

1 = {s : s is equinumerous with 0}

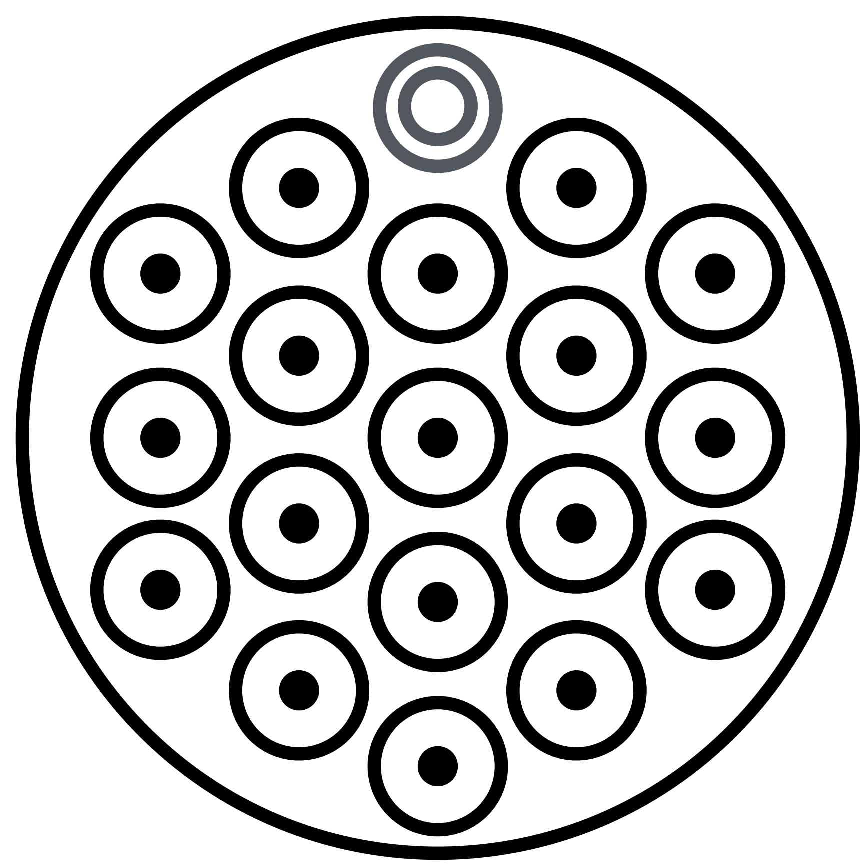
$1 = \{s : s \text{ is equinumerous with } \{\{x : x \neq x\}\}\ \}$



1 = the extension of the concept: is equinumerous with the concept: is identical to 0

1 = {s : s is equinumerous with 0}

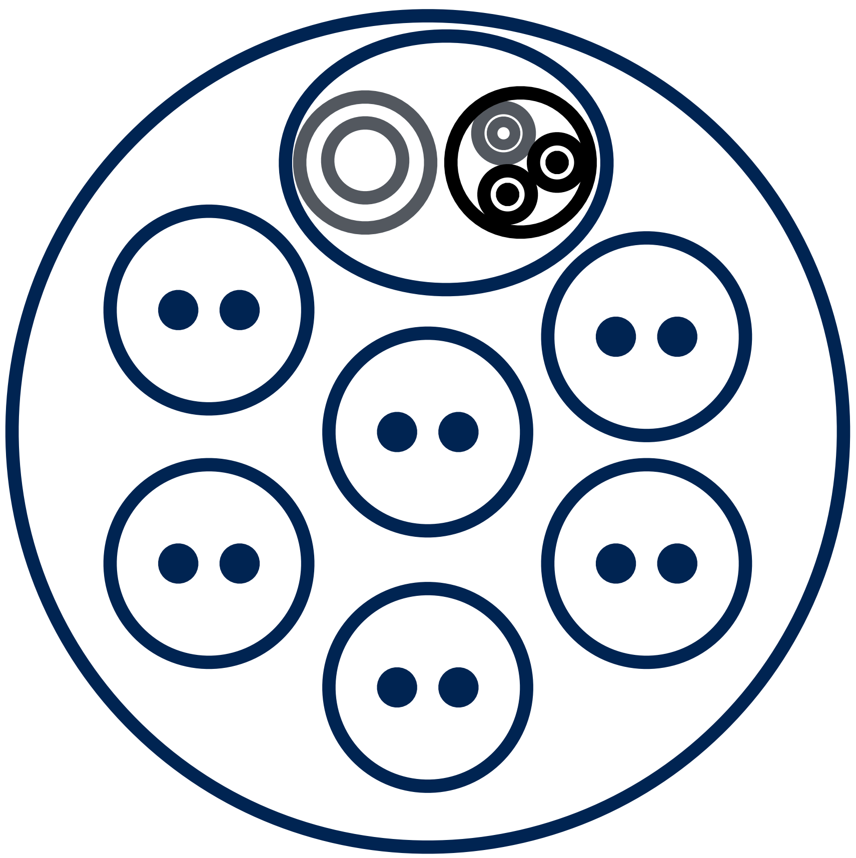
1 = {s : s is equinumerous with {{x : $x \neq x$}} }



2 = the extension of the concept: is equinumerous with the concept: is identical to 0 or is identical to 1

$2 = \{s : s \text{ is equinumerous with } \{x : x{=}0 \vee x{=}1\}\}$

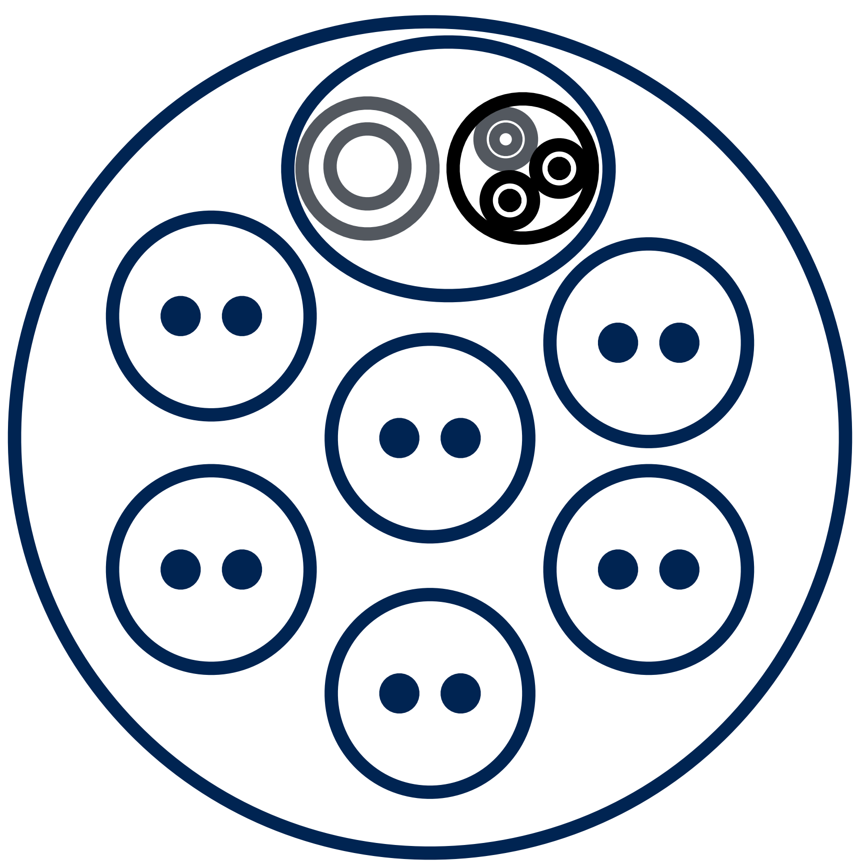
$2 = \{s : s \text{ is equinumerous with } \{0, 1\}$

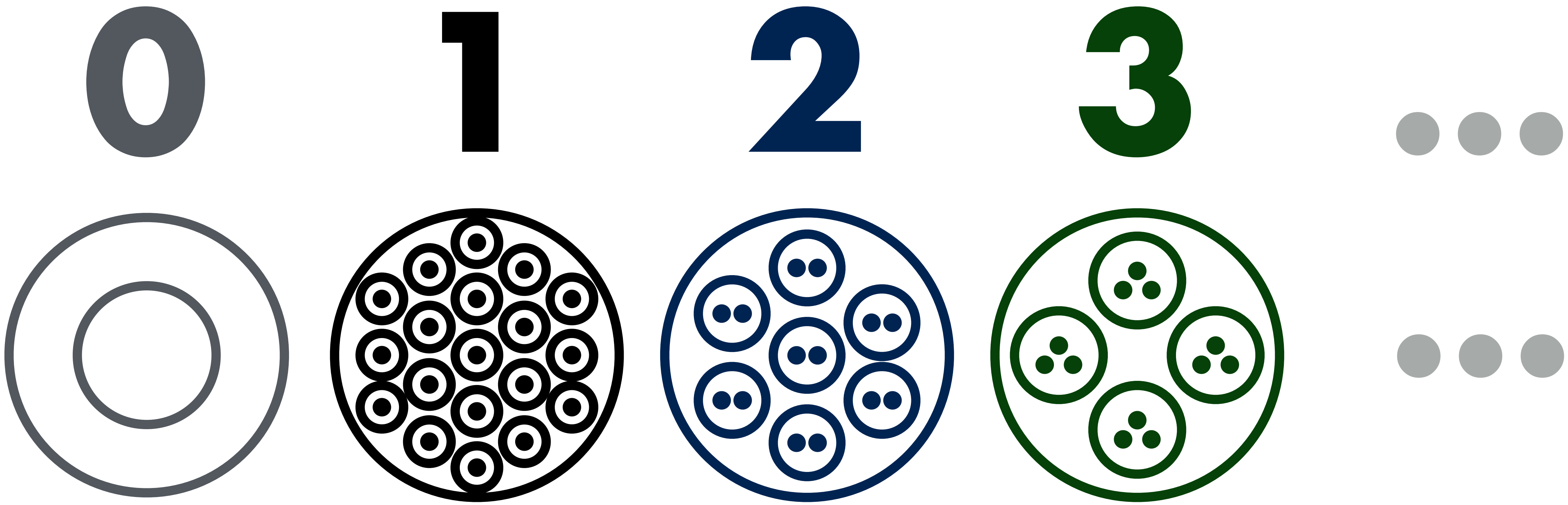


3 = the extension of the concept: is equinumerous with the concept: is identical to 0 or is identical to 1 or is identical to 2

3 = {s : s is equinumerous with $\{x : x=0 \vee x=1 \vee x=2\}$}

3 = {s : s is equinumerous with {0, 1, 2}}





- Good for counting.
- Satisfies the Peano postulates, and so has the right structure for doing higher mathematics.

Puzzling Facts about Mathematical Truths

Necessity
It's not just a matter of fact that 2+2=4. It apparently could not have been otherwise. ✅

A priority
We seem to be able to know mathematical truths in a pure, not wholly sensory way. ✅

Applicability
Mathematics is a very useful guide to how the world actually works. ✅

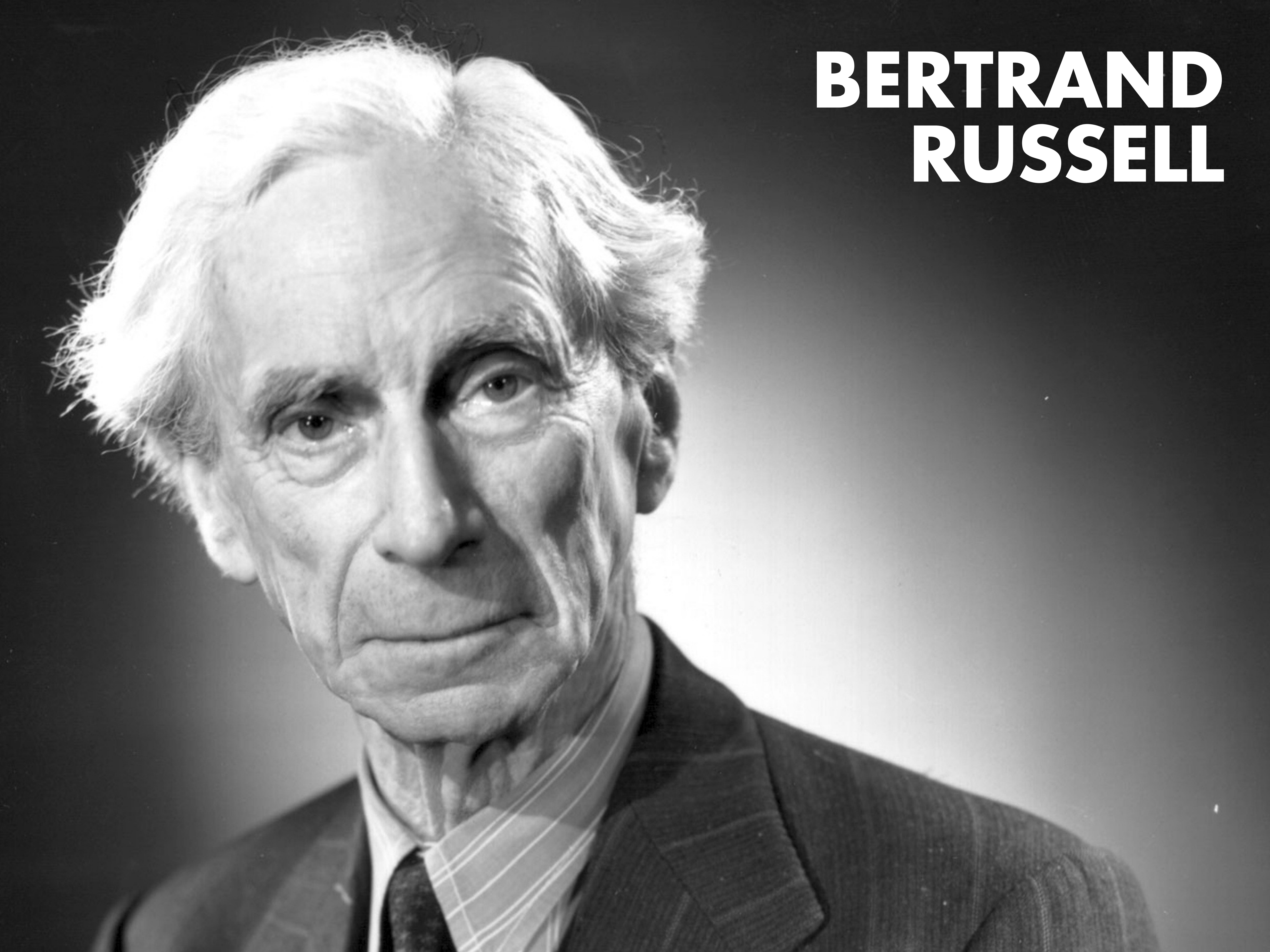
Theories about how we know these facts

Logicism
Mathematical knowledge is all derivable from logical truths, which are the most general and abstract truths about the world—so general and abstract that they seem trivial.

Wait so what are numbers?

Whether I would say that the reduction of higher mathematics to arithmetics say we don't have to posit the ontological existence of numbers …. well I don't know, there is a way in which descriptions still beg the question of what numbers are ontologically — but I just might need more clarification on how numbers are being proved ontological through reduction to logic. Particularly as explained that definitions of number come through successors and zero "Rather, we may define 2 as the successor of 1, which in turn may be defined as the successor of zero." (144)

—Monika


BERTRAND
RUSSELL

FREGE'S BASIC LAW 5

For any concepts, *F* and *G*, the extension of *F* is identical to the extension of *G* if and only if for every object *a*, *Fa* if and only if *Ga*.

A CONSEQUENCE/PRESUPPOSITION

Every concept F has an extension.

RUSSELL'S PARADOX (v1)

"Let *w* be the predicate: to be a predicate that cannot be predicated of itself. Can *w* be predicated of itself? From each answer the opposite follows. Therefore must conclude that *w* is not a predicate."

—Russell, 1902 letter to Frege

(Here, 'predicate' means something like what Frege means by 'concept'.)

RUSSELL'S PARADOX (v2)

"Likewise, there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves."

—Russell, 1902 letter to Frege

Russell’s Paradox

I'm confused by the first claim of Russel's paradox. The paradox is a paradox because of the fact that the infinite set is a set of all and only those things that are not members of themselves. What exactly does Russel mean by something being a member of itself — why would something be a member of itself to begin with?

—Boaz

EXPLOSION

$P \land \neg P \vdash Q$

(1) $P \land \neg P$	Premise
(2) P	1, $\land$-elimination
(3) $P \lor Q$	2, $\lor$-introduction
(4) $\neg P$	1, $\land$-elimination
(5) Q	3, 4, $\lor$-elimination

"Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic. … It is all the more serious since, with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish."

—Frege, 1902 Letter to Russell

As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.

—Russell, in a letter to Jean van Heijenoort, years later

Theorem: Every natural number is interesting.

Proof: "Suppose that not every natural number is interesting. Then the set of uninteresting natural numbers is non-empty. So by the well-ordering property of the natural numbers, it must have a smallest element n. But if n is the smallest uninteresting natural number, then n is interesting for that very reason. Thus we have a contradiction, establishing that our original hypothesis was false, and that every natural number is interesting after all."

—John Burgess, 'Tarski's Tort'

“the…usual reaction to this bit of adolescent mathematical humor is that ‘interesting’ is too vague or ambiguous, too subjective or relative, a concept to be admissible in mathematical reasoning.”

—John Burgess, ‘Tarski’s Tort’

THE VICIOUS-CIRCLE PRINCIPLE

"An analysis of the paradoxes to be avoided shows that they all result from a kind of vicious circle. The vicious circles in question arise from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole. Thus, for example, the collection of propositions will be supposed to contain a proposition stating that "all propositions are either true or false." It would seem, however, that such a statement could not be legitimate unless "all propositions" referred to some already definite collection, which it cannot do if new propositions are created by statements about "all propositions." We shall, therefore, have to say that statements about "all propositions" are meaningless."

—Whitehead and Russell, *Principia Mathematica*

THE VICIOUS-CIRCLE PRINCIPLE

"The principle which enables us to avoid illegitimate totalities may be stated as follows: "Whatever involves all of a collection must not be one of the collection"; or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total." We shall call this the "vicious-circle principle," because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities."

—Whitehead and Russell, *Principia Mathematica*

How to understand the theory of types?

The most interesting part is the way to phrase this hierarchy of subscript sets that resolves the russell's paradox. i.e. "the universe is structured such that..." or "The limits of language are such that we can only describe the universe in a certain way" (154) and how both of these ideas seem flawed, might the truth of this matter be where Wittgenstein threw his chalk into the air and said, "we must not speak of it"?

—Wes

THE THEORY OF TYPES (ROUGHLY)

The world is organized into the following infinite hierarchy:

Type 0: individuals (things that aren't sets)

Type 1: sets of individuals

Type 2: sets of individuals and type-1 sets

Type 3: sets of individuals and type-1 or type-2 sets

Type 4: sets of individuals and sets of types 1–3

etc.

THE THEORY OF TYPES (ROUGHLY)

Our languages organized into a hierarchy:

Type 0: names of individuals

Type 1: predicates of individuals

Type 2: predicates of type-1 predicates (and individuals)

Type 3: predicates of things that are type 0–2.

Type 4: predicates of things that are type 0–3.

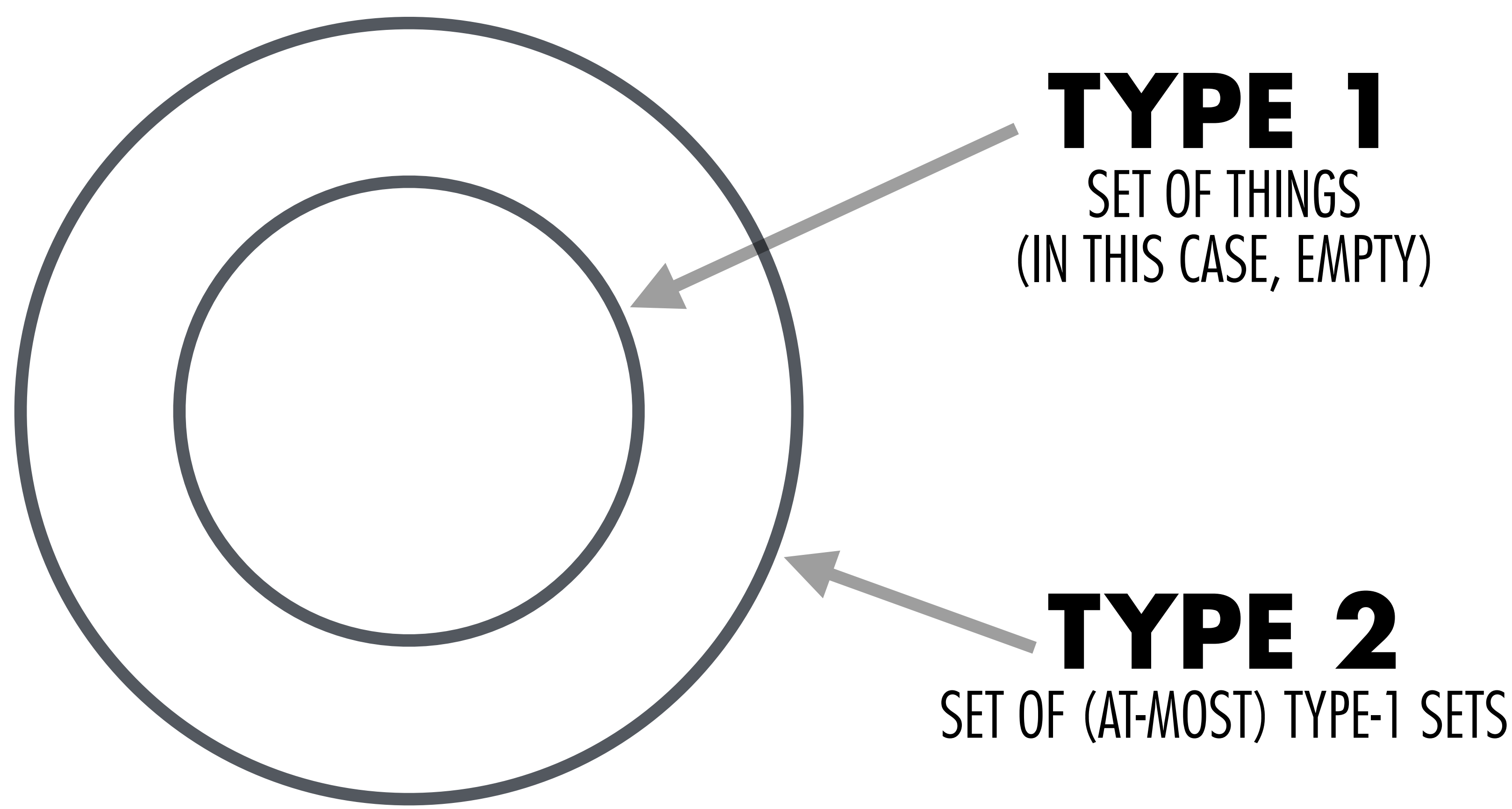
etc.

A general principle of our language:

Any sentence in which a predicate F is predicated of something x is meaningful only if the type of x is lesser than the type of F.

0 = the extension of the concept: is equinumerous with the concept: is not self-identical

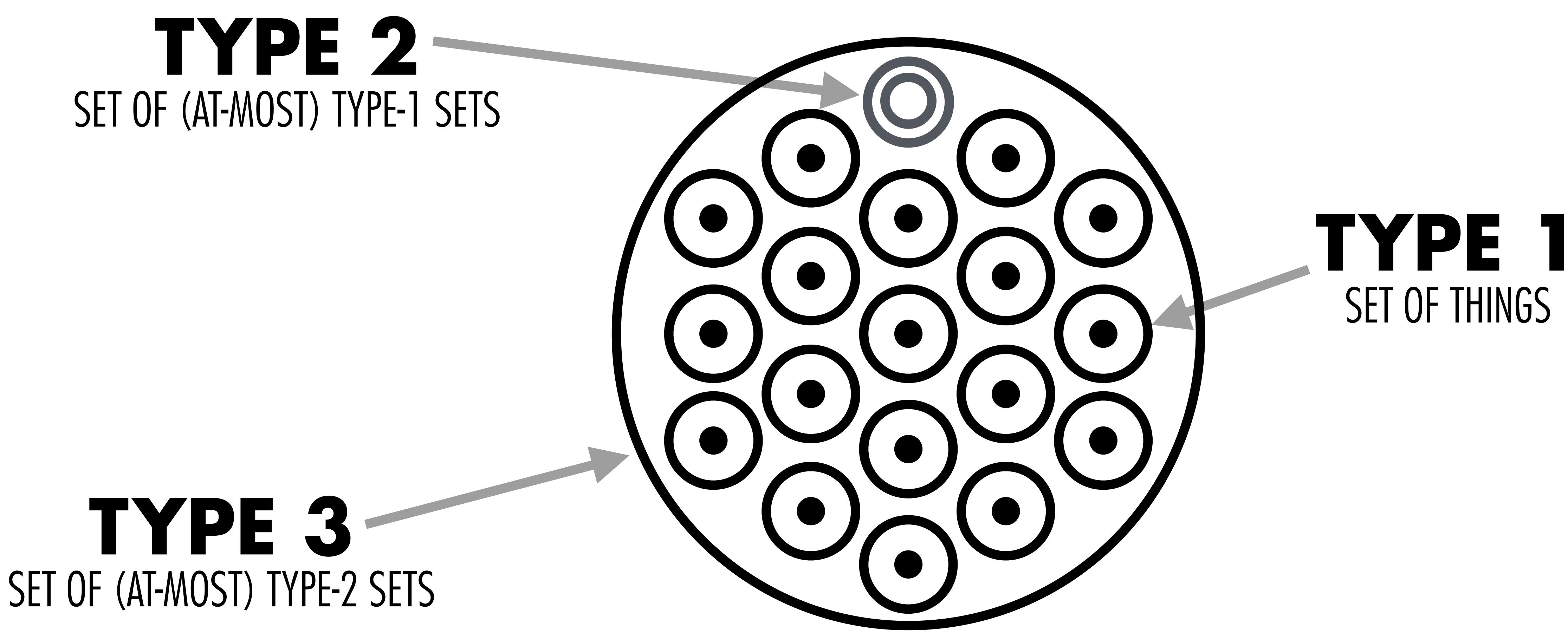
0 = {s : s is equinumerous with $\{x : x \neq x\}$}



1 = the extension of the concept: is equinumerous with the concept: is identical to 0

1 = {s : s is equinumerous with 0}

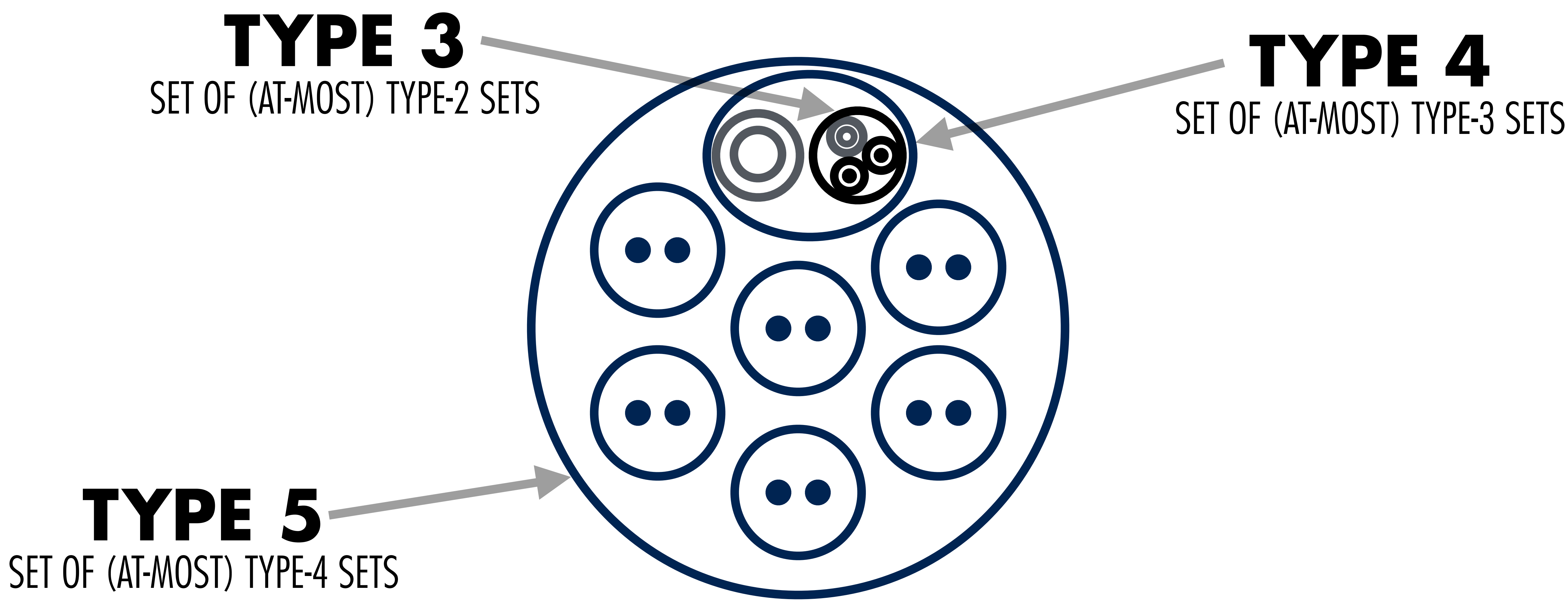
1 = {s : s is equinumerous with $\{\{x : x \neq x\}\}$ }



2 = the extension of the concept: is equinumerous with the concept: is identical to 0 or is identical to 1

$2 = \{s : s \text{ is equinumerous with } \{x : x{=}0 \lor x{=}1\}\}$

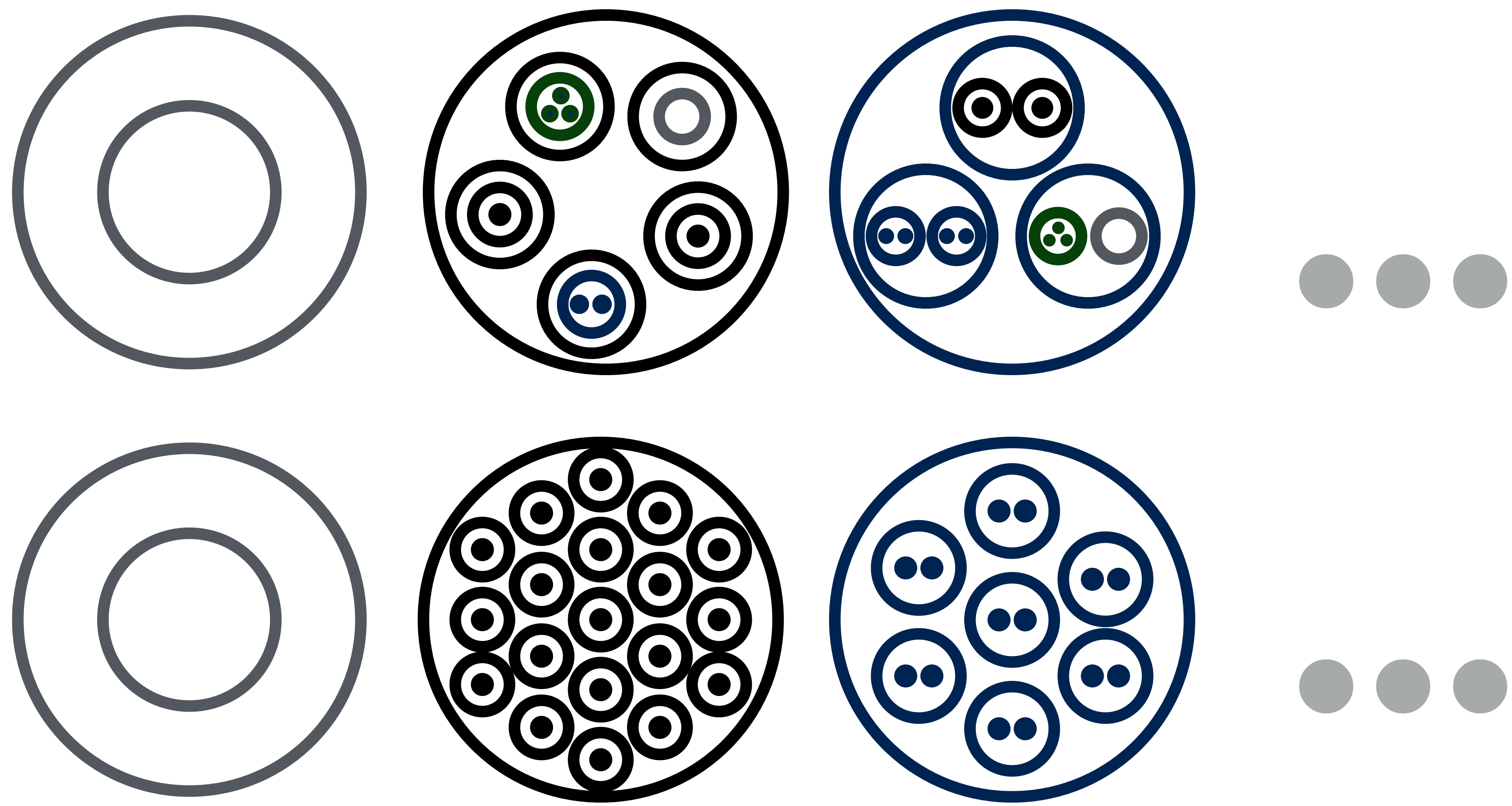
$2 = \{s : s \text{ is equinumerous with } \{0, 1\}$



NUMBERS OF DIFFERENT TYPES

TYPE 3:
(SETS OF SETS OF SETS OF THINGS)

TYPE 2:
(SETS OF SETS OF THINGS)



THE AXIOM OF INFINITY

Roughly: There are an infinite number of (type-0) individuals.

THE AXIOM OF REDUCIBILITY

"Thus a predicative function of an individual is a first-order function; and for higher types of arguments, predicative functions take the place that first-order functions take in respect of individuals. We assume then, that every function is equivalent, for all its values, to some predicative function of the same argument. This assumption seems to be the essence of the usual assumption of classes [modern sets] . . . we will call this assumption the axiom of classes, or the axiom of reducibility."

Speculative Foundations

Pertaining to the recurring notion of speculative definitions or starting points used for reduction and also many times in deductive argumentation, I'm wondering what's the reason for this speculation. Is it because of the infinite regress of reason, or something else?

—Tayyab

Russell and Gödel

Is it fair to say that perhaps the most important result of the Principia Mathematica was that it provided a basis for Godel's proof that you cannot prove all of the truths of arithmetic within a consistent theoretical system? In other words, as best I understand it, apart from any other problems that there might be with the *Principia Mathematica*, Godel proved that it was impossible to create a set of axioms from which you could logically derive all "true" arithmetical statements without generating any logical contradictions.

—Seth