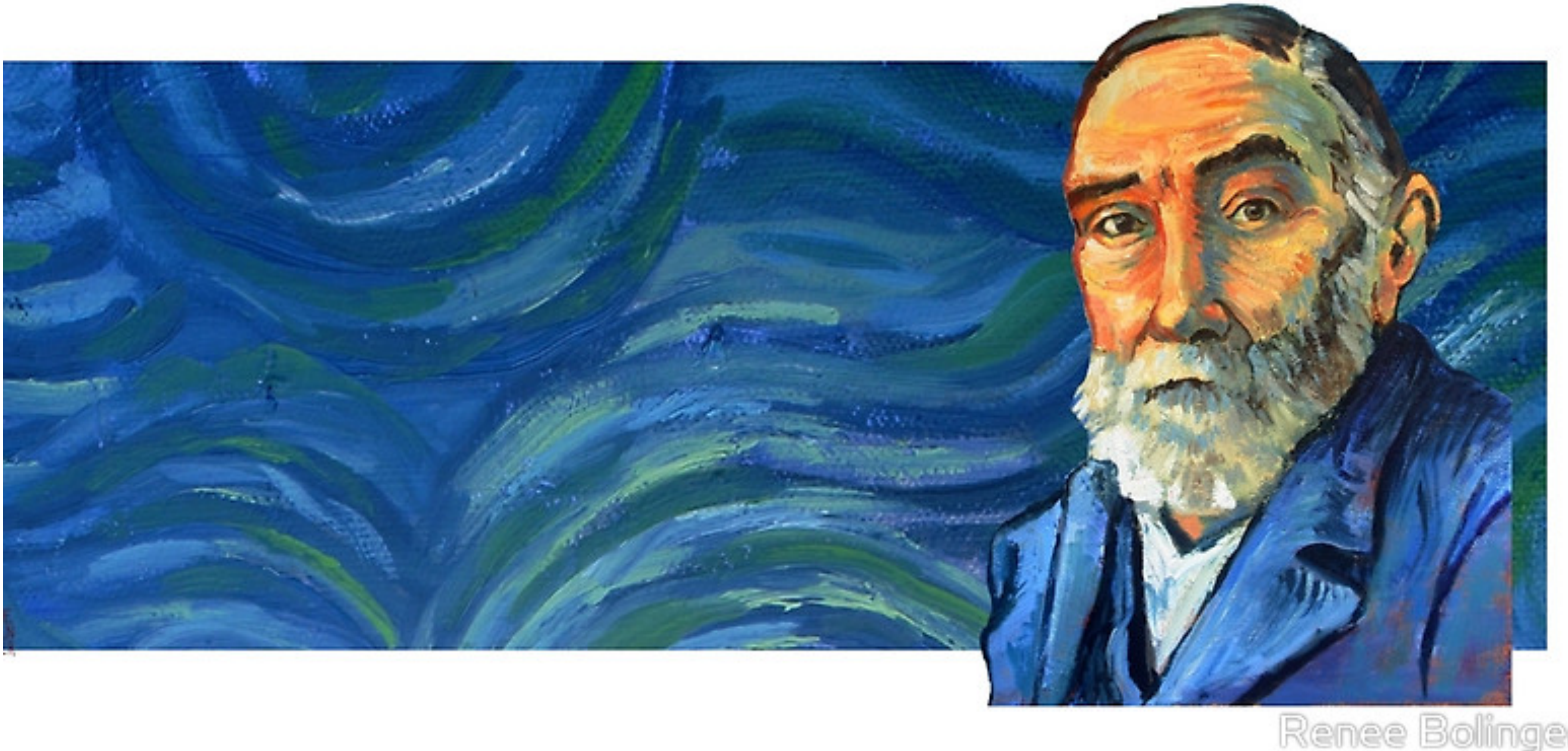
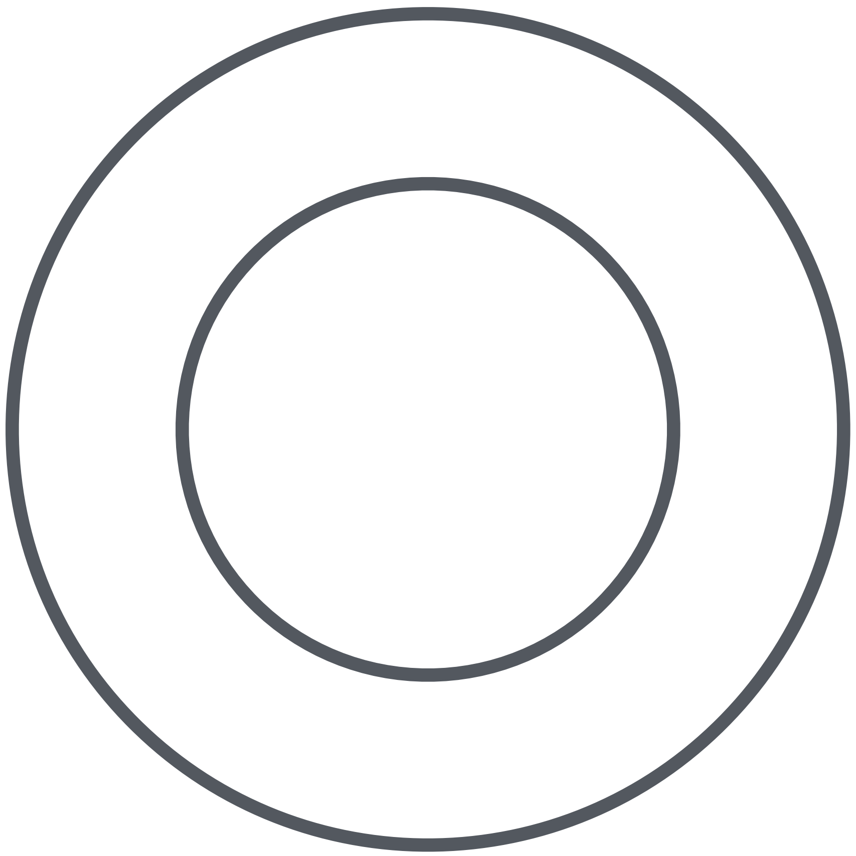
GOTTLOB FREGE

THE FOUNDATIONS OF ARITHMETIC



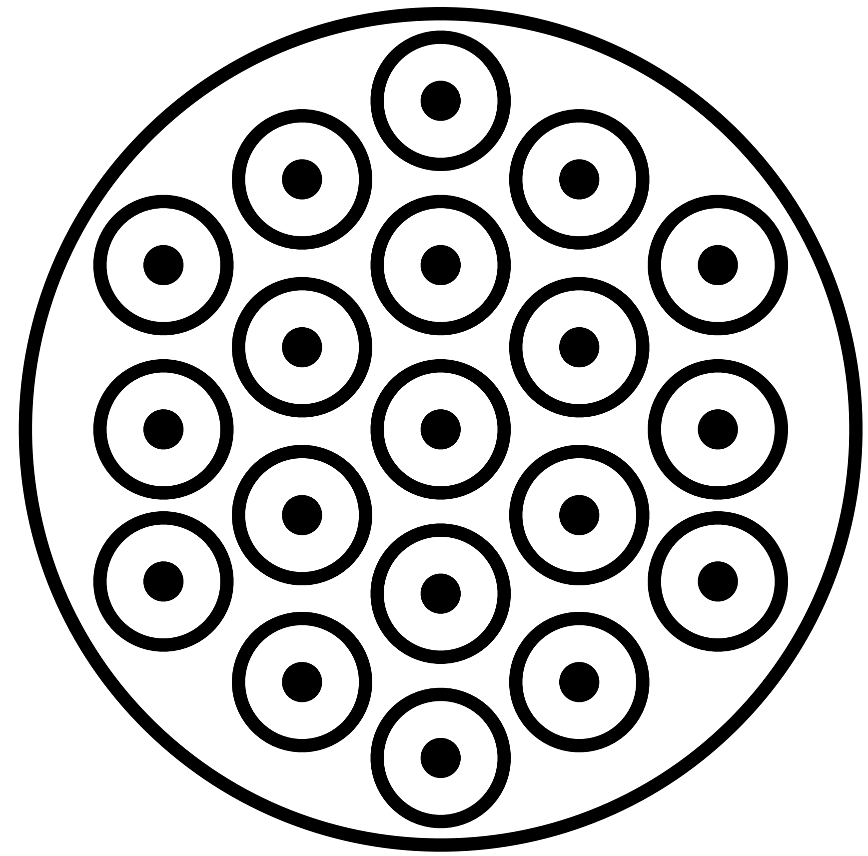
0



0 is the set of all sets containing no members

$$0 =_{df} \{s : \neg(\exists x)(x \in s)\}$$

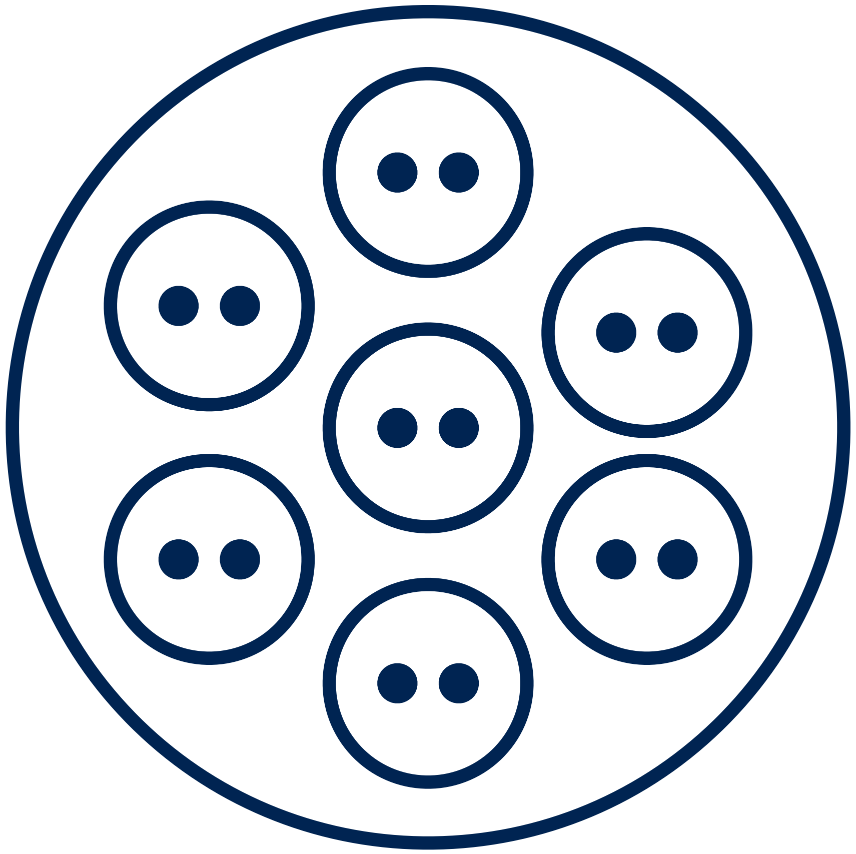
1



1 is the set of all sets containing a single member

$$1 =_{df} \{s : (\exists x)(x \in s \ \& \ (\forall y)(y \in s \supset y = x))\}$$

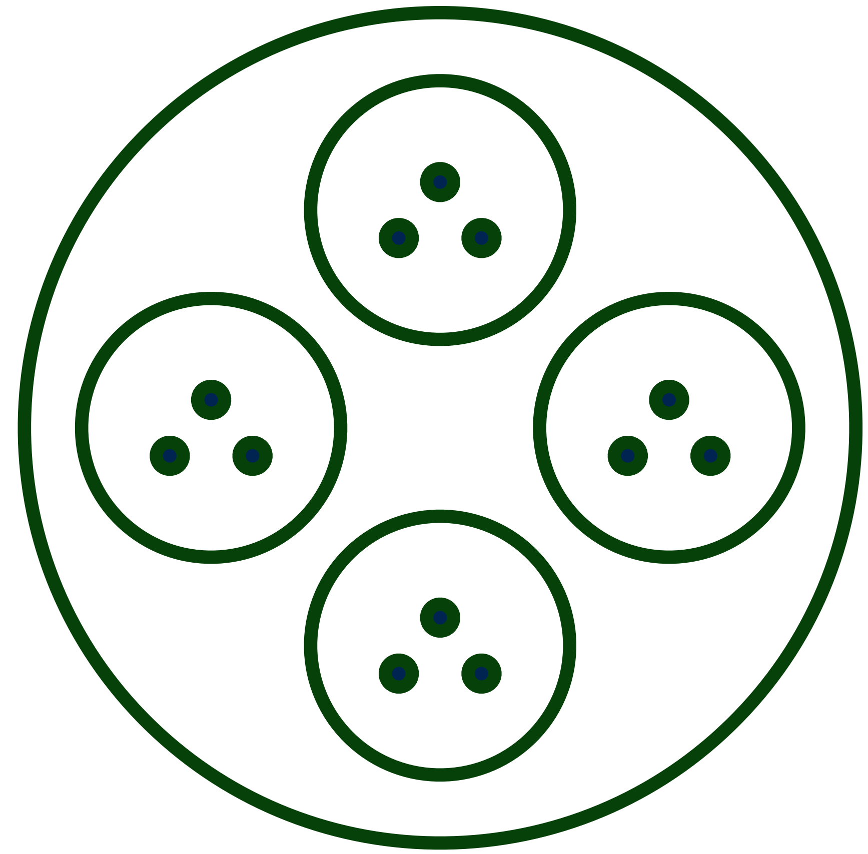
2



2 is the set of all sets that have exactly two members

$$=_{df} \{s : (\exists x)(\exists y)(x \in s \;\&\; y \in s \;\&\; x \neq y \;\&\; (\forall z)(z \in s \supset (z \neq x \supset z = y))\}$$

3



3 is the set of all sets containing exactly three members

$$=_{df} \{s : (\exists x)(\exists y)(\exists z)(x \in s \;\&\; y \in s \;\&\; z \in s \;\&\; x \neq y \;\&\; x \neq z \;\&\; y \neq z \;\&\; (\forall w)(w \in s \supset (w \neq x \supset (w \neq y \supset w = z))))\}$$

PEANO POSTULATES

(1) 0 is a number

(2) The successor of any number is a number

(3) No two numbers have the same successor

(4) 0 is not the successor of any number

(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.

ADDITION

(i) $a + 0 = a$

(ii) $a + S(b) = S(a+b)$

$a + 1$	$= a + S(0)$	(by the def. of 1)
	$= S(a + 0)$	(by (ii))
	$= S(a)$	(by (i))

$a + 2$	$= a + S(1)$	(by the def. of 2)
	$= S(a + 1)$	(by (ii))
	$= S(S(a))$	(by the result of a+1)

0	1	2	3	...
100	101	102	104	...
1	3	5	7	...
1	1/2	1/4	1/8	...

0	1	2	3	...
100	101	102	104	...
1	3	5	7	...
1	1/2	1/4	1/8	...

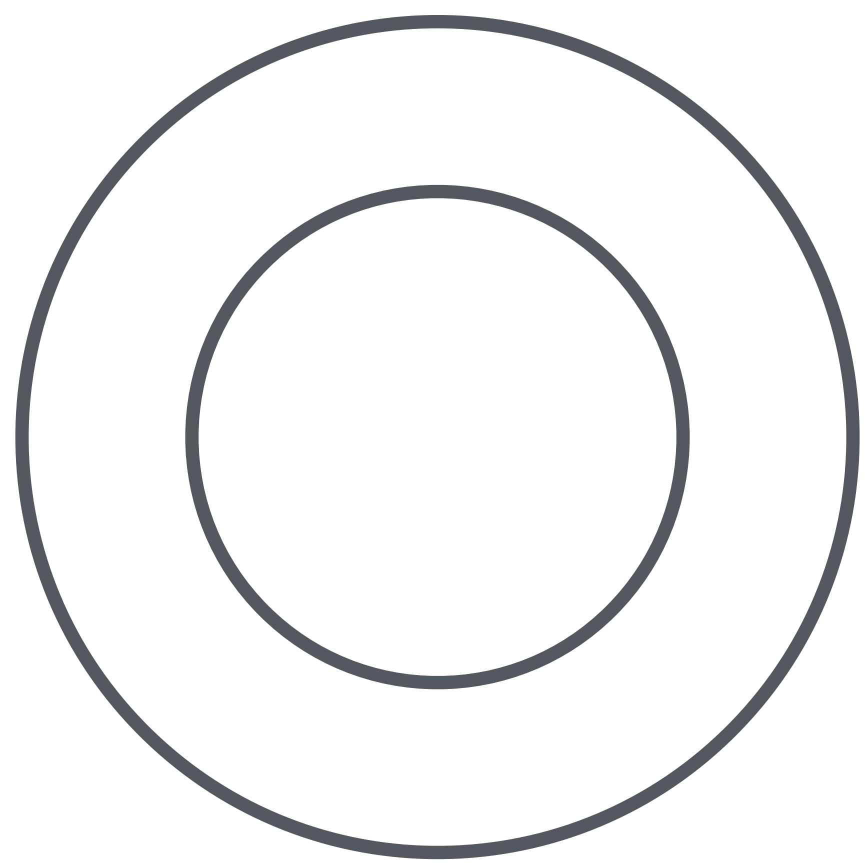
The number that belongs to the concept F =
the number that belongs to the concept G

iff

There is a one-to-one correspondence
between the Fs and the Gs.

0 = the extension of the concept: is equinumerous with the concept: is not self-identical

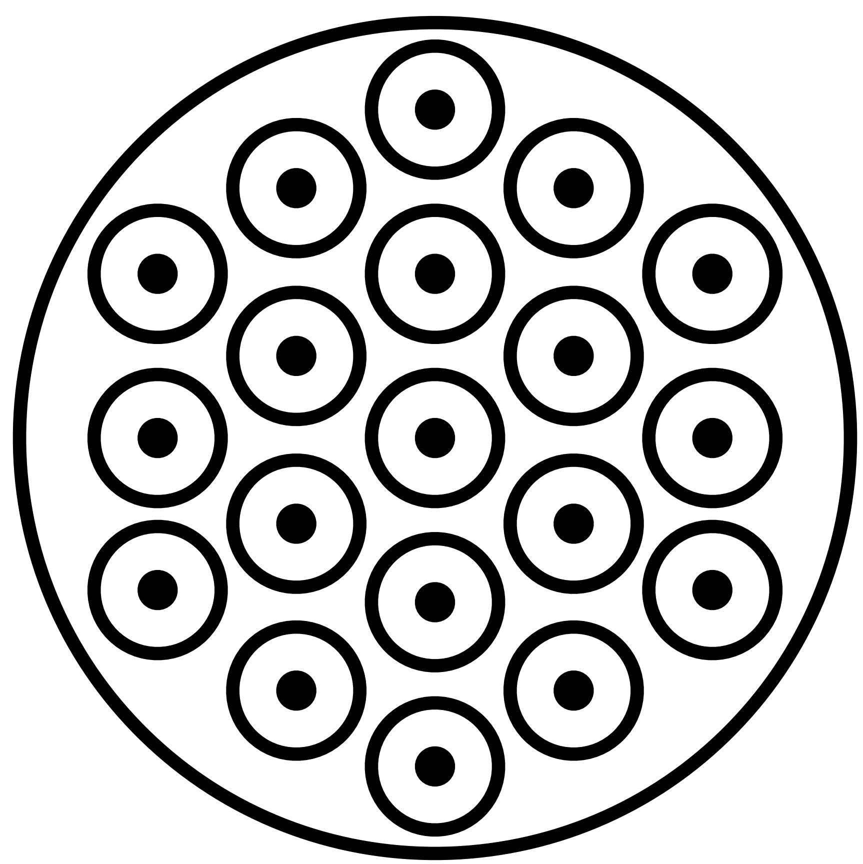
0 = {s : s is equinumerous with $\{x : x \neq x\}$}



1 = the extension of the concept: is equinumerous with the concept: is identical to 0

1 = {s : s is equinumerous with 0}

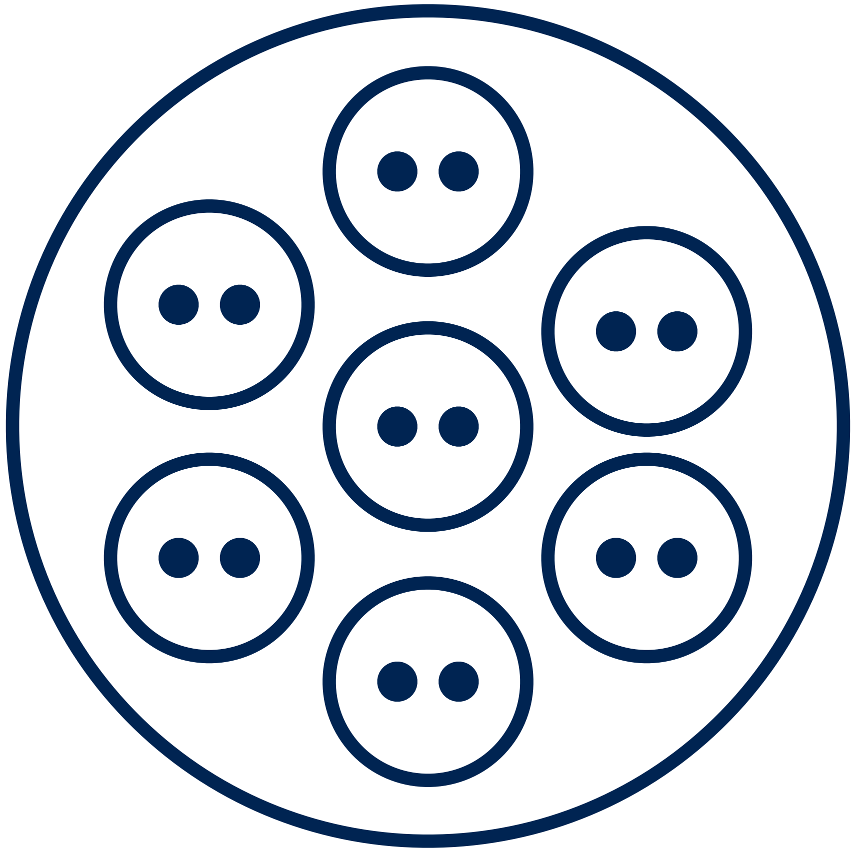
1 = {s : s is equinumerous with $\{\{x : x \neq x\}\}$ }



2 = the extension of the concept: is equinumerous with the concept: is identical to 0 or is identical to 1

$2 = \{s : s \text{ is equinumerous with } \{x : x=0 \lor x=1\}\}$

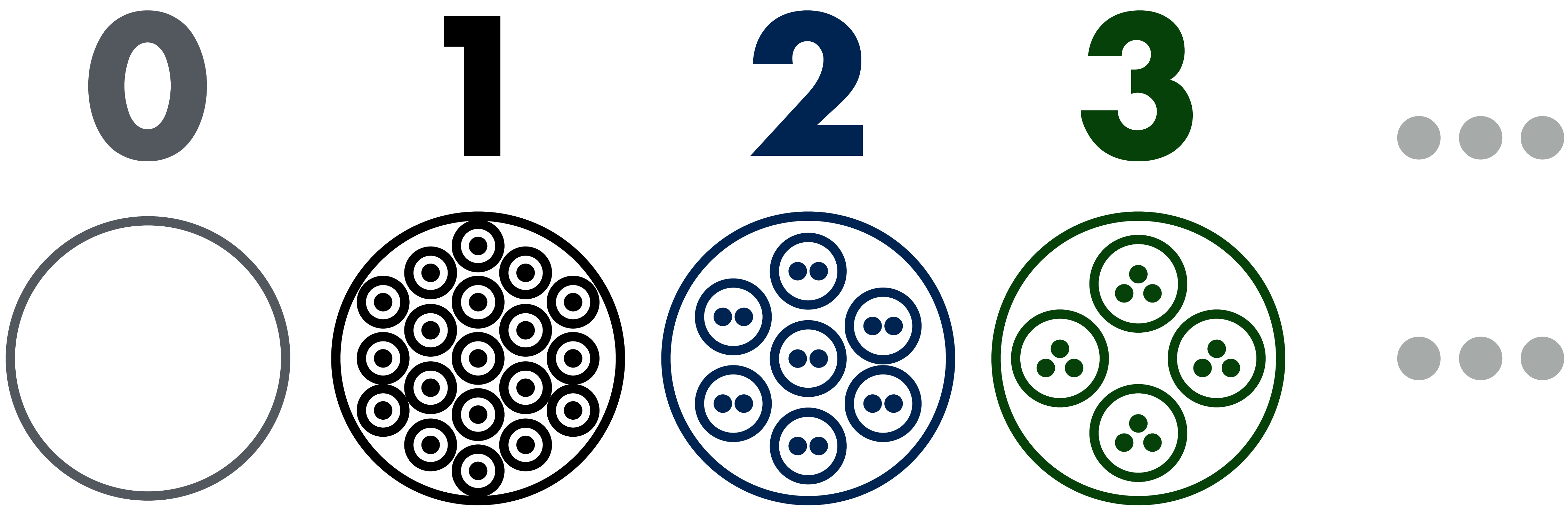
$2 = \{s : s \text{ is equinumerous with } \{\varnothing, \{\varnothing\}\}\ \}$



SUCCESSOR RELATION

For any two numbers, *m* and *n*, *n* is the successor of *m* if and only if:

There is a concept F and an object falling under it *x* such that: the number which belongs to F is *n* and the number which belongs to the concept *falling under F but not identical with x* is m.


0
1
2
3