Background to Gottlob Frege
Gottlob Frege
(1848–1925)

Life’s work: logicism
(the reduction of arithmetic to logic).

This entailed:

• Inventing (discovering?) modern logic, including quantification, variables, etc.

• Investigating the properties of the language in which this logic was spelled out.
Step 1: *Begriffsschrift* (1879)

...in which Frege comes up with "classical" logic.
What is Logic?

Deductive logic is the attempt to formulate a precise theory that predicts which conclusions follow from which premises.

These predictions are based solely on the structures of the sentences involved.
Syllogistic Logic

From Aristotle until the 19th Century, almost all logic was syllogistic. E.g.:

- All men are mortals.
- All mortals will die.
- So, all men will die.

- All A are B.
- All B are C.
- So, all A are C.

Syllogistic logic assumes that sentences are structured into subjects and predicates.

Frege showed that this is a bad way to do things.
The Problem with Subject-Predicate Structure

It can’t help us to predict certain good inferences. E.g.:

John loves Steve.  
So John loves someone.  

John loves Steve.  
So Steve is loved by someone.
Frege on Sentence Structure

Frege pointed out that there are multiple ways to break sentences down into parts:

\[ \text{John loves } \_ \_ \_ + \text{Steve} + \_ \_ \_ \text{loves Steve} + \text{John} + \_ \_ \_ \text{loves } \_ \_ \_ \_ + \text{John} + \text{Steve} \]

The “unsaturated” parts are predicates. They refer to “concepts” (Begriffe). The “saturated” parts (names) refer to “objects”.

The “extension” of a concept is the set of all the things that the concept is true of.
Frege on Sentence Structure

Frege pointed out that there are multiple ways to break sentences down into parts:

John loves ___ + Steve   ___ loves Steve + John   ___ loves ___ ___ + John + Steve

The “unsaturated” parts are predicates. They refer to “concepts” (Begriffe). The “saturated” parts (names) refer to “objects”.

Thus the name that Frege gave to his logic: “concept script” (Begriffsschrift)
Functions and Arguments
Frege argued that we should think of concepts as functions (in the mathematical sense).

They take objects as their arguments and map the to truth values (truth or falsity).

John loves ___(Steve) = truth       John loves ___(Donald) = falsity
Quantifiers

What about the structure of a quantificational sentence, like these?

- John loves everyone
- Someone loves Steve
- Everyone loves someone

Quantifier words, like “everyone” and “someone”, don’t look like they refer to individual objects.

Frege argued that they refer to second-level concepts. Concepts that take first-level concepts as arguments and map them to truth values:

\[
\text{everyone}(\text{____ loves Steve})
\]

(the concept of being being true of everyone applies to the concept of loving Steve)
Notation

English: Everything is good.

The notation I’ve been using: everything( ___ is good )

Frege’s Notation: a G(a)

Contemporary logical notation: (x)Gx or: (\forall x)Gx
English: Everything good is not bad.

Frege’s Notation:

Contemporary logical notation: $(\forall x)(Gx \rightarrow \neg Bx)$
Step 2: *Die Grundlagen der Arithmetik* (1884)

...in which Frege proposes his definition of number.
\[0 = \{s : \neg(\exists x)(x \in s)\}\]

0 is the set of all sets containing no members

\[1 = \{s : (\exists x)(x \in s \land (\forall y)(y \in s \iff y = x))\}\]

1 is the set of all sets containing a single member
2 is the set of all sets that have exactly two members

\[=_{df}\{s: (\exists x)(\exists y)(x \in s \& \ y \in s \& \ x \neq y \ \& \ (\forall z)(z \in s \subset (z \neq x \cup z = y))\}\]

3 is the set of all sets containing exactly three members

\[=_{df}\{s: (\exists x)(\exists y)(\exists z)(x \in s \& \ y \in s \& \ z \in s \ \& \ x \neq y \ \& \ x \neq z \ \& \ y \neq z \ \& \ (\forall w)(w \in s \subset (w \neq x \cup (w \neq y \cup w = z))))\}\]
Step 3: *Die Grundgesetze der Arithmetik* (1893–1903)

...in which Frege attempts to derive arithmetic from logic
The number that belongs to the concept $F$ = the number that belongs to the concept $G$

iff

There is a one-to-one correspondence between the Fs and the Gs.
$0 = \text{the extension of the concept: is equinumerous with the concept: is not self-identical}$

$0 = \{s : s \text{ is equinumerous with } \{x : x \neq x\}\}$
1 = the extension of the concept: is equinumerous with the concept: is identical to 0

1 = \{ s : s \text{ is equinumerous with } 0 \}

1 = \{ s : s \text{ is equinumerous with } \{ \{ x : x \neq x \} \} \}
$2 = \text{the extension of the concept: is equinumerous with the concept: is identical to 0 or is identical to 1}$

$2 = \{ s : s \text{ is equinumerous with } \{ x : x=0 \lor x=1 \} \}$

$2 = \{ s : s \text{ is equinumerous with } \{ 0, 1 \} \}$
PEANO POSTULATES

(1) 0 is a number

(2) The successor of any number is a number

(3) No two numbers have the same successor

(4) 0 is not the successor of any number

(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.
FREGE’S BASIC LAW 5
For any concepts, $F$ and $G$, the extension of $F$ is identical to the extension of $G$ if and only if for every object $a$, $Fa$ if and only if $Ga$.

A CONSEQUENCE/PRESUPPOSITION
Every concept $F$ has an extension.
Step 2.5: Three Philosophical Papers

_Funktion und Begriff_ (1891)
(‘Function and Concept’)  
...in which Frege proposes that concepts (the meanings of predicates) should be thought of as _functions_.

_Über Begriff und Gegenstand_ (1892)
(‘Concept and Object’)  
...in which Frege argues that concepts/functions differ from objects in that they’re "unsaturated".

_Über Sinn und Bedeutung_ (1892)
(‘Sense and Reference’)  
...in which Frege elaborates his theory of words’ meanings.
The Problem:

“π” and “3.14159265359…” refer to the same object
I.e., these two symbols have the same meaning.

So then why do we learn something when we find out that

\[ \pi = 3.14159265359... \]

After all, we can’t learn anything from \[ \pi = \pi \]

But we’ve only substituted synonyms for synonyms here.
"It is natural, now, to think of there being connected with a sign (name, combination of words, letter), besides that to which the sign refers, which may be called the reference of the sign, also what I should like to call the sense of the sign, wherein the mode of presentation is contained."
Each expression can have (at least) the following two semantic properties:

- **sense** \((\text{Sinn})\)
- **reference** \((\text{Bedeutung})\)

*(Frege also mentions a third kind in passing: **coloring** \((\text{Färbung})\).*
Frege's Puzzle:

Compare:

(1) Hesperus is identical to Hesperus.

(2) Hesperus is identical to Phosphorus.

and:

(3) The morning star is identical to the morning star.

(4) The morning star is identical to the evening star.
Frege’s Puzzle (in general):

Identity statements of the form ‘$a=a$’ are trivial, but those of the form ‘$a=b$’ are nontrivial. We get information from the second kind but not the first kind. They differ in cognitive value.

If the meaning of an expression is just its referent, we can’t explain this.
Frege's Solution to his Puzzle:

“If we found "a=a" and "a=b" to have different cognitive values, the explanation is that for the purpose of knowledge, the sense of the sentence, viz., the thought expressed by it, is no less relevant than its referent, i.e., its truth value. If now a=b, then indeed the referent of "b" is the same as that of "a," and hence the truth value of "a=b" is the same as that of "a=a." In spite of this, the sense of "b" may differ from that of "a," and thereby the sense expressed in "a=b" differs from that of "a=a." In that case the two sentences do not have the same cognitive value.”
Job description for sense:

- It is responsible for the expression’s "cognitive value"—i.e., what you can learn from sentences containing the expression.
Frege's Anti-Psychologism

“The sense of a proper name is grasped by everybody who is sufficiently familiar with the language or totality of designations to which it belongs; but this serves to illuminate only a single aspect of the referent, supposing it to exist. Comprehensive knowledge of the referent would require us to be able to say immediately whether every given sense belongs to it. To such knowledge we never attain.”
Frege’s Anti-Psychologism

"The reference and sense of a sign are to be distinguished from the associated idea. ... The idea is subjective: one man’s idea is not another. ... This constitutes an essential difference between the idea and the sign’s sense, which may be the common property of many and therefore is not a part of a mode of the individual mind." (8th paragraph)
Frege’s Anti-Psychologism

"By a thought I understand not the subjective performance of thinking but its objective content, which is capable of being the common property of several thinkers." (fn.7)
The Sense of a Sentence is a Thought

"By a thought I understand not the subjective performance of thinking but its objective content, which is capable of being the common property of several thinkers." (fn.7)
Thoughts and Truth-Conditions

The sense of a sentence is “the thought that its truth-conditions are fulfilled”.

(Frege, Grundgesetzte, §32)
Job description for sense:

- It is responsible for the expression’s "cognitive value"—i.e., what you can learn from sentences containing the expression.
- It is the objective informational content that is communicated to others by means of the expression.
The Sense as Mode of Presentation of the Referent

“A difference can arise only if the difference between the signs corresponds to a difference in the mode of presentation of that which is designated. Let a, b, c be the lines connecting the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of a and b is then the same as the point of intersection of b and c. So we have different designations for the same point, and these names ("Point of intersection of a and b," “Point of intersection of b and c") likewise indicate the mode of presentation; and hence the statement contains true knowledge.”
Job description for sense:

• It is responsible for the expression’s "cognitive value"—i.e., what you can learn from sentences containing the expression.

• It is the objective informational content that is communicated to others by means of the expression.

• It is a mode of presentation of the expression’s referent
Sense as Indirect Reference

“In reported speech one talks about the sense—e.g., of another person's remarks. It is quite clear that in this way of speaking words do not have their customary referents but designate what is usually their sense. In order to have a short expression, we will say: In reported speech, words are used indirectly or have their indirect referents.”
Sense as Indirect Reference

We can normally substitute co-referring expressions without changing the reference of expressions in which they appear.

e.g.:

**Hesperus** is identical to Venus.

**Phosphorus** is identical to Venus.
Sense as Indirect Reference

Some syntactic positions don’t allow substitution of co-referring expressions.

Lucy believes that **Hesperus** is identical to Venus.

Lucy believes that **Phosphorus** is identical to Venus.
Sense as Indirect Reference

Frege’s solution to this problem is that expressions in these "oblique" (ungerade) contexts refer to their usual senses.

Lucy believes that Hesperus is identical to Venus.

Lucy believes that Phosphorus is identical to Venus.
Indirect Contexts

Anything embedded under a "propositional attitude verb"

Lois believes that Superman is not Clark Kent.

Lois desires superman.

Lois said that superman is brave.

Lois went outside in order to see Superman.
Job description for sense:

- It is responsible for the expression’s "cognitive value"—i.e., what you can learn from sentences containing the expression.
- It is the objective informational content that is communicated to others by means of the expression.
- It is a mode of presentation of the expression’s referent
- It is the referent of the expression in "indirect" (ungerade) linguistic contexts.
Job description for sense:

- It is responsible for the expression’s "cognitive value"—i.e., what you can learn from sentences containing the expression.
- It is the objective informational content that is communicated to others by means of the expression.
- It is a mode of presentation of the expression’s referent
- It is the referent of the expression in "indirect" (ungerade) linguistic contexts.
- It determines the expression’s referent.
“An object can be determined in different ways, and every one of these ways of determining it can give rise to a special name, and these different names then have different senses; for it is not self-evident that it is the same object which is being determined in different ways.”

Sentences Refer to Truth Values

Why does Frege think this?

pp.214–215
<table>
<thead>
<tr>
<th>SIGN</th>
<th>NAME</th>
<th>PREDICATE</th>
<th>SENTENCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>a linguistic expression</td>
<td>‘Aristotle’</td>
<td>‘...is wise’</td>
<td>‘Aristotle is wise.’</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>SENSE</th>
<th>a mode of presentation of an individual</th>
<th>a mode of presentation of a concept</th>
<th>A THOUGHT</th>
</tr>
</thead>
<tbody>
<tr>
<td>what is expressed by the sign</td>
<td>the student of Plato and teacher of Alexander</td>
<td>our way of thinking about the concept of wisdom (whatever that is)</td>
<td>An abstract informational content. Can be either true, false, or neither, and can be grasped by different agents</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>REFERENCE</th>
<th>AN INDIVIDUAL</th>
<th>A CONCEPT</th>
<th>TRUTH VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>what the sign designates</td>
<td>Aristotle himself</td>
<td>a function from things to truth values</td>
<td>either truth or falsity</td>
</tr>
</tbody>
</table>

|                |                          | the concept of wisdom: a function that wise things to truth and non-wise things to false | (in this case) truth |