SYMBOLIC LOGIC

UNIT 7:
THE PROOF
METHOD: 8 BASIC
INFERENCE RULES

Constants vs. Variables

$A \vee B$

Constants are abbreviations of specific sentences with determinate meanings.

$p \vee q$

Variables are placeholders for any formula whatever, including both simple and complex formulas.

Substitution Instances

A substitution instance (s.i.) of a statement form is a statement obtained by substituting (uniformly) some statement for each variable in the statement form.

We must substitute the same statement for repeated occurrences of the same variable, and we may substitute the same statement for different variables.

Thus both A v B and A ∨ A are s.i.'s of p ∨ q, but A ∨ B is not an s.i. of p ∨ p.

Modus Ponens (M.P.)

$$\begin{array}{l} p \supset q \\ p \\ \hline /\therefore q \end{array}$$

Modus Tollens (M.T.)

$$\begin{array}{l} p \supset q \\ \sim q \\ \hline /\therefore \sim p \end{array}$$

Hypothetical Syllogism (H. S.)

$$\begin{array}{l} p \supset q \\ q \supset r \\ \hline /\therefore p \supset r \end{array}$$

Simplification (Simp.)

$$\begin{array}{l} p \cdot q \\ \hline /\therefore p \end{array} \qquad \begin{array}{l} p \cdot q \\ \hline /\therefore q \end{array}$$

Conjunction (Conj.)

$$\begin{array}{l} p \\ q \\ \hline /\therefore p \cdot q \end{array}$$

Dilemma (Dil.)

$$\begin{array}{l} p \supset q \\ r \supset s \\ p \vee r \\ \hline /\therefore q \vee s \end{array}$$

Disjunctive Syllogism (D. S.)

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline /\therefore q \end{array} \qquad \begin{array}{l} p \vee q \\ \sim q \\ \hline /\therefore p \end{array}$$

Addition (Add.)

$$\begin{array}{l} p \\ \hline /\therefore p \vee q \end{array} \qquad \begin{array}{l} q \\ \hline /\therefore p \vee q \end{array}$$

6. For each of the following arguments, construct a proof of the conclusion from the premises, using only the rules of M.P., M.T., and H.S. Be sure to justify every step.

b. $(T \vee W) \supset A$, $(C \supset \sim B)$, $(A \supset C)$, $\sim \sim B$ / $\therefore \sim (T \vee W)$

7. Construct proofs for the following, using only the rules for the conditional and conjunction.

d. $(A \vee B) \supset (D \cdot C)$, $\sim C \supset \sim (D \cdot C)$, $\sim (A \vee B) \supset \sim A$, $(\sim C \cdot \sim F)$ $/ \therefore \sim A \cdot \sim F$

7. Construct proofs for the following, using only the rules for the conditional and conjunction.

d. $(A \vee B) \supset (D \cdot C)$, $\sim C \supset \sim (D \cdot C)$, $\sim (A \vee B) \supset \sim A$, $(\sim C \cdot \sim F)$ $/ \therefore \sim A \cdot \sim F$

8. For each of the following arguments, construct a proof of the conclusion from the given premises, and justify every step that is not a premise. These and all following proofs may use any of the eight basic rules of inference.

a. $D \supset (A \vee C), D \cdot \sim A \quad / \therefore C$

8. For each of the following arguments, construct a proof of the conclusion from the given premises, and justify every step that is not a premise. These and all following proofs may use any of the eight basic rules of inference.

r. $(A \vee B) \supset (C \vee D), (C \supset E), (C \vee \sim F), (A \cdot \sim E), (F \vee (D \supset Z))$ $/ \therefore Z$

SYMBOLIC LOGIC
UNIT 8:
REPLACEMENT
RULES

DN: $p :: \sim\sim p$

Dup: $p :: p \vee p$
$p :: p \bullet p$

Comm: $p \vee q :: q \vee p$
$p \bullet q :: q \bullet p$

Assoc: $(p \vee q) \vee r :: p \vee (q \vee r)$
$(p \bullet q) \bullet r :: p \bullet (q \bullet r)$

Contra: $p \supset q :: \sim q \supset \sim p$

DeM: $\sim(p \vee q) :: \sim p \bullet \sim q$
$\sim(p \bullet q) :: \sim p \vee \sim q$

BE: $p \equiv q :: (p \supset q) \bullet (q \supset p)$

CE: $p \supset q :: \sim p \vee q$

Dist: $p \bullet (q \vee r) :: (p \bullet q) \vee (p \bullet r)$
$p \vee (q \bullet r) :: (p \vee q) \bullet (p \vee r)$

Exp: $(p \bullet q) \supset r :: p \supset (q \supset r)$

DN: p :: ~~p

Dup: p :: p∨p
p :: p•p

Comm: p∨q :: q∨p
p•q :: q•p

Assoc: (p∨q)∨r :: p∨(q∨r)
(p•q)•r :: p•(q•r)

Contra: p⊃q :: ~q⊃~p

DeM: ~(p∨q) :: ~p•~q
~(p•q) :: ~p∨~q

BE: p≡q :: (p⊃q)•(q⊃p)

CE: p⊃q :: ~p∨q

Dist: p•(q∨r) :: (p•q)∨(p•r)
p∨(q•r) :: (p∨q)•(p∨r)

Exp: (p•q)⊃r :: p⊃(q⊃r)

Unit 8, #5b

1. ~(A ∨ B) Pr. /∴ ~B

DN: p :: ~~p

Dup: p :: p∨p
p :: p•p

Comm: p∨q :: q∨p
p•q :: q•p

Assoc: (p∨q)∨r :: p∨(q∨r)
(p•q)•r :: p•(q•r)

Contra: p⊃q :: ~q⊃~p

DeM: ~(p∨q) :: ~p•~q
~(p•q) :: ~p∨~q

BE: p≡q :: (p⊃q)•(q⊃p)

CE: p⊃q :: ~p∨q

Dist: p•(q∨r) :: (p•q)∨(p•r)
p∨(q•r) :: (p∨q)•(p∨r)

Exp: (p•q)⊃r :: p⊃(q⊃r)

Unit 8, #5c

1. ~(A ⊃ B) Pr. / ∴ ~B

DN: p :: ~~p

Dup: p :: p∨p
p :: p•p

Comm: p∨q :: q∨p
p•q :: q•p

Assoc: (p∨q)∨r :: p∨(q∨r)
(p•q)•r :: p•(q•r)

Contra: p⊃q :: ~q⊃~p

DeM: ~(p∨q) :: ~p•~q
~(p•q) :: ~p∨~q

BE: p≡q :: (p⊃q)•(q⊃p)

CE: p⊃q :: ~p∨q

Dist: p•(q∨r) :: (p•q)∨(p•r)
p∨(q•r) :: (p∨q)•(p∨r)

Exp: (p•q)⊃r :: p⊃(q⊃r)

Unit 8, #5f

1.	A ⊃ B	Pr.
2.	A ⊃ C	Pr. /∴ A ⊃ (B • C)

Unit 8, #5m

1. A ⊃ C   Pr. /∴ (A • B) ⊃ C

DN: p :: ~~p

Dup: p :: p∨p
p :: p•p

Comm: p∨q :: q∨p
p•q :: q•p

Assoc: (p∨q)∨r :: p∨(q∨r)
(p•q)•r :: p•(q•r)

Contra: p⊃q :: ~q⊃~p

DeM: ~(p∨q) :: ~p•~q
~(p•q) :: ~p∨~q

BE: p≡q :: (p⊃q)•(q⊃p)

CE: p⊃q :: ~p∨q

Dist: p•(q∨r) :: (p•q)∨(p•r)
p∨(q•r) :: (p∨q)•(p∨r)

Exp: (p•q)⊃r :: p⊃(q⊃r)

DN: p :: ~~p

Dup: p :: p∨p
p :: p•p

Comm: p∨q :: q∨p
p•q :: q•p

Assoc: (p∨q)∨r :: p∨(q∨r)
(p•q)•r :: p•(q•r)

Contra: p⊃q :: ~q⊃~p

DeM: ~(p∨q) :: ~p•~q
~(p•q) :: ~p∨~q

BE: p≡q :: (p⊃q)•(q⊃p)

CE: p⊃q :: ~p∨q

Dist: p•(q∨r) :: (p•q)∨(p•r)
p∨(q•r) :: (p∨q)•(p∨r)

Exp: (p•q)⊃r :: p⊃(q⊃r)

Unit 8, #5n

1. ~((A ∨ B) ∨ (C ∨ D)) Pr. /∴ ~D

DN: p :: ~~p

Dup: p :: p∨p
p :: p•p

Comm: p∨q :: q∨p
p•q :: q•p

Assoc: (p∨q)∨r :: p∨(q∨r)
(p•q)•r :: p•(q•r)

Contra: p⊃q :: ~q⊃~p

DeM: ~(p∨q) :: ~p•~q
~(p•q) :: ~p∨~q

BE: p≡q :: (p⊃q)•(q⊃p)

CE: p⊃q :: ~p∨q

Dist: p•(q∨r) :: (p•q)∨(p•r)
p∨(q•r) :: (p∨q)•(p∨r)

Exp: (p•q)⊃r :: p⊃(q⊃r)

Unit 8, #5o

1.	A	Pr.
2.	~B	Pr. /∴ ~(A ≡ B)

DN: p :: ~~p

Dup: p :: p∨p
p :: p•p

Comm: p∨q :: q∨p
p•q :: q•p

Assoc: (p∨q)∨r :: p∨(q∨r)
(p•q)•r :: p•(q•r)

Contra: p⊃q :: ~q⊃~p

DeM: ~(p∨q) :: ~p•~q
~(p•q) :: ~p∨~q

BE: p≡q :: (p⊃q)•(q⊃p)

CE: p⊃q :: ~p∨q

Dist: p•(q∨r) :: (p•q)∨(p•r)
p∨(q•r) :: (p∨q)•(p∨r)

Exp: (p•q)⊃r :: p⊃(q⊃r)

Unit 8, #5p

1. ~A ⊃ A Pr. /∴ A

DN: p :: ~~p

Dup: p :: pvp
p :: p•p

Comm: pvq :: qvp
p•q :: q•p

Assoc: (pvq)vr :: pv(qvr)
(p•q)•r :: p•(q•r)

Contra: p⊃q :: ~q⊃~p

DeM: ~(pvq) :: ~p•~q
~(p•q) :: ~pv~q

BE: p≡q :: (p⊃q)•(q⊃p)

CE: p⊃q :: ~pvq

Dist: p•(qvr) :: (p•q)v(p•r)
pv(q•r) :: (pvq)•(pvr)

Exp: (p•q)⊃r :: p⊃(q⊃r)

Unit 8, #6k

1.	(P • G) ⊃ R	Pr.
2.	(R • S) ⊃ T	Pr.
3.	P • S	Pr.
4.	G ∨ R	Pr. /∴ R ∨ T

DN: $p :: {\sim}{\sim}p$

Dup: $p :: p \vee p$
$p :: p \bullet p$

Comm: $p \vee q :: q \vee p$
$p \bullet q :: q \bullet p$

Assoc: $(p \vee q) \vee r :: p \vee (q \vee r)$
$(p \bullet q) \bullet r :: p \bullet (q \bullet r)$

Contra: $p \supset q :: {\sim}q \supset {\sim}p$

DeM: ${\sim}(p \vee q) :: {\sim}p \bullet {\sim}q$
${\sim}(p \bullet q) :: {\sim}p \vee {\sim}q$

BE: $p \equiv q :: (p \supset q) \bullet (q \supset p)$

CE: $p \supset q :: {\sim}p \vee q$

Dist: $p \bullet (q \vee r) :: (p \bullet q) \vee (p \bullet r)$
$p \vee (q \bullet r) :: (p \vee q) \bullet (p \vee r)$

Exp: $(p \bullet q) \supset r :: p \supset (q \supset r)$

Unit 8, #7a

1. $({\sim}A \vee {\sim}B) \supset {\sim}C$ Pr.
2. $(A \supset F)$ Pr.
3. ${\sim}F \equiv (D \bullet {\sim}E)$ Pr.
4. ${\sim}(D \supset H)$ Pr.
5. $E \supset H$ Pr. / ∴ ${\sim}C$

DN: p :: ~~p

Dup: p :: p∨p
p :: p•p

Comm: p∨q :: q∨p
p•q :: q•p

Assoc: (p∨q)∨r :: p∨(q∨r)
(p•q)•r :: p•(q•r)

Contra: p⊃q :: ~q⊃~p

DeM: ~(p∨q) :: ~p•~q
~(p•q) :: ~p∨~q

BE: p≡q :: (p⊃q)•(q⊃p)

CE: p⊃q :: ~p∨q

Dist: p•(q∨r) :: (p•q)∨(p•r)
p∨(q•r) :: (p∨q)•(p∨r)

Exp: (p•q)⊃r :: p⊃(q⊃r)

Unit 8, #7f

1. ~P ≡ ~(Q ⊃ R) Pr.
2. ~(P ∨ (S ∨ T)) Pr.
3. Z ⊃ W Pr.
4. ~(R ∨ T) ⊃ ~(S • W) Pr. / ∴ ~Z

DN:	$p :: \sim\sim p$
Dup:	$p :: p \vee p$
	$p :: p \bullet p$
Comm:	$p \vee q :: q \vee p$
	$p \bullet q :: q \bullet p$
Assoc:	$(p \vee q) \vee r :: p \vee (q \vee r)$
	$(p \bullet q) \bullet r :: p \bullet (q \bullet r)$
Contra:	$p \supset q :: \sim q \supset \sim p$
DeM:	$\sim(p \vee q) :: \sim p \bullet \sim q$
	$\sim(p \bullet q) :: \sim p \vee \sim q$
BE:	$p \equiv q :: (p \supset q) \bullet (q \supset p)$
CE:	$p \supset q :: \sim p \vee q$
Dist:	$p \bullet (q \vee r) :: (p \bullet q) \vee (p \bullet r)$
	$p \vee (q \bullet r) :: (p \vee q) \bullet (p \vee r)$
Exp:	$(p \bullet q) \supset r :: p \supset (q \supset r)$

SYMBOLIC LOGIC

UNIT 9: CONDITIONAL PROOF AND INDIRECT PROOF

CONDITIONAL PROOF (C.P.)

p
.
.
q

$/ \therefore p \supset q$

If, given the assumption p, we are able to derive q, then we are allowed to infer $(p \supset q)$, citing all the steps from p to q inclusive.

INDIRECT PROOF (I.P.)

p
.
.
.
$q \cdot \sim q$

$/ \therefore \sim p$

If, given an assumption p, we are able to derive a contradiction $(q \cdot \sim q)$, then we may infer the negation of our assumption, $\sim p$, citing all the steps from p to $(q \cdot \sim q)$ inclusive.

a.

1.	$(A \vee B) \supset (C \cdot D)$	Pr. / $\therefore A \supset C$
2.	→ A	Assp. (C.P.)
3.	$A \vee B$	Add. 2
4.	$C \cdot D$	M.P. 1,3
5.	C	Simp. 4
6.	$A \supset C$	C.P. 2–5

c.

1.	$A \supset (B \supset C)$	Pr. / $\therefore (\sim B \supset \sim A) \supset (A \supset C)$
2.	→ $\sim B \supset \sim A$	Assp. (C.P.)
3.	→ A	Assp. (C.P.)
4.	$A \supset B$	Contrap. 2
5.	B	M.P. 3,4
6.	$B \supset C$	M.P. 1,3
7.	C	M.P. 5,6
8.	$A \supset C$	C.P. 3–7
9.	$(\sim B \supset \sim A) \supset (A \supset C)$	C.P. 2–8

CONDITIONAL PROOF (C.P.)

→ p
.
.
q

/ $\therefore p \supset q$

If, given the assumption p, we are able to derive q, then we are allowed to infer $(p \supset q)$, citing all the steps from p to q inclusive.

d.

1.	$N \supset O$	Pr.
2.	$(N \cdot O) \supset P$	Pr.
3.	$P \supset \sim O$	Pr. / $\therefore \sim N$
4.	$\rightarrow N$	Assp. (I.P.)
5.	O	M.P. 1,4
6.	$N \cdot O$	Conj. 4,5
7.	P	M.P. 2,6
8.	$\sim O$	M.P. 3,7
9.	$O \cdot \sim O$	Conj. 5,8
10.	$\sim N$	I.P. 4–9

INDIRECT PROOF (I.P.)

$\rightarrow p$

.

.

.

$q \cdot \sim q$

$/ \therefore \sim p$

If, given an assumption p, we are able to derive a contradiction $(q \cdot \sim q)$, then we may infer the negation of our assumption, $\sim p$, citing all the steps from p to $(q \cdot \sim q)$ inclusive.

c.	1.	$A \supset (B \supset C)$	Pr. $/ \therefore (\sim B \supset \sim A) \supset (A \supset C)$
	2.	$\sim B \supset \sim A$	Assp. (C.P.)
	3.	A	Assp. (C.P.)
	4.	$A \supset B$	Contrap. 2
	5.	B	M.P. 3,4
	6.	$B \supset C$	M.P. 1,3
	7.	C	M.P. 5,6
	8.	$A \supset C$	C.P. 3–7
	9.	$(\sim B \supset \sim A) \supset (A \supset C)$	C.P. 2–8

SUBPROOF

A section of a proof starting with an assumption and finishing when the assumption is discharged.

THREE RULES ABOUT SUBPROOFS

1. All assumptions must be discharged before the end of a proof.
2. Once a subproof is finished (after the assumption is discharged), none of the lines of the subproof may be used in later justifications.
3. Subproof lines can't cross: when more than one subproof is happening at the same time, the most recent assumption must be discharged first.

4. Construct proofs for the following, using the rule of C.P. plus the rules from Units 7 and 8.

4b 1. $(\sim A \vee \sim B) \supset \sim C$ Pr. $/\therefore C \supset A$

2.
3.
4.
5.
6.
7.
8.
9.
10.

5. Construct proofs for the following, using the rule of I.P. plus the rules from Units 7 and 8.

5b 1. A • ~B Pr. /∴ ~(A ≡ B)

2.

3.

4.

5.

6.

7.

8.

9.

10.

6d

1	A ≡ ~(B ∨ C)	Pr.
2	B ≡ (D • ~E)	Pr.
3	~(E • A)	Pr /∴ A ⊃

1	
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Unit 9, Exercise 7: Construct a proof of the following theorems.

h. $(p \supset ({\sim}p \supset q))$

1.
2.
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4.
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11.
12.
13.
14.
15.
16.

Unit 9, Exercise 7: Construct a proof of the following theorems.

b. $(p \supset (p \bullet q)) \vee (q \supset (p \bullet q))$

1.
2.
3.
4.
5.
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10.
11.
12.
13.
14.
15.
16.

Unit 9, Exercise 7: Construct a proof of the following theorems.

m. $p \equiv (p \vee (q \cdot {\sim}q))$

1.
2.
3.
4.
5.
6.
7.
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9.
10.
11.
12.
13.
14.
15.
16.

Unit 9, Exercise 7: Construct a proof of the following theorems.

n. $p \equiv (p \bullet (q \vee {\sim}q))$

1.
2.
3.
4.
5.
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7.
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10.
11.
12.
13.
14.
15.
16.